Ordinary Differential Equations

Notes from the text "Differential Equations and Boundary Value Problems: Computing and Modeling" by C. H. Edwards and D. E. Penney

Jeffrey A. Oregero
Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA
January 22, 2025

Contents

1 Introduction:

Loosely speaking a differential equation is an equation between an unknown function and it's derivatives. The **order** of a differential equation is the order of the highest derivative appearing in the equation. **Solving a differential equation means finding all functions that satisfy the equation**.

(i) If the unknown is a function of a single independent variable we say that the differential equation is an **ordinary differential equation**, or simply, an **ODE**.

(ii) If the unknown is a function of multiple independent variables we say that the differential equation is a **partial differential equation**, or simply, a **PDE**.

Definition 1.1. The general n^{th}-order ODE with independent variable t and unknown, or dependent variable, $y = y(t)$ is given by

$$F(t, y, y', y'', \dots, y^{(n)}) = 0, \tag{1.1}$$

where F is a specific real-valued function of $n + 2$ variables.

Definition 1.2. We say that the continuous function $u = u(t)$ is a **solution** of the differential equation (1.1) on an interval I provided that the derivatives $u', u'', \dots, u^{(n)}$ exist on I and

$$F(t, u, u', u'', \dots, u^{(n)}) = 0$$

for all $t \in I$.

Example 1.1. Consider the function $t \mapsto y(t)$ defined by $y(t) = (C - t)^{-1}$ where C is an arbitrary constant. Then it is easy to see

$$y'(t) = -(C - t)^{-2}(-1) = \left(\frac{1}{C - t}\right)^2 = y(t)^2.$$

Thus, $y(t) = (C - t)^{-1}$ is a **one-parameter family of solutions** to the differential equation $y' = y^2$. Here "one-parameter family of solutions" means that we have a different solution for each value of the constant C (see Fig. 1).

In this course we will ordinarily assume that the differential equation can be solved explicitly for the highest derivative that appears; that is, the equation can be written in the so-called **normal**, or **explicit**, form

$$y^{(n)} = G(t, y, y', y'', \dots, y^{(n-1)}), \tag{1.2}$$

where G is a real-valued function of $n + 1$ variables. Indeed, we will usually exploit the functional form of G in order to identify an appropriate method to solve the ODE under consideration.

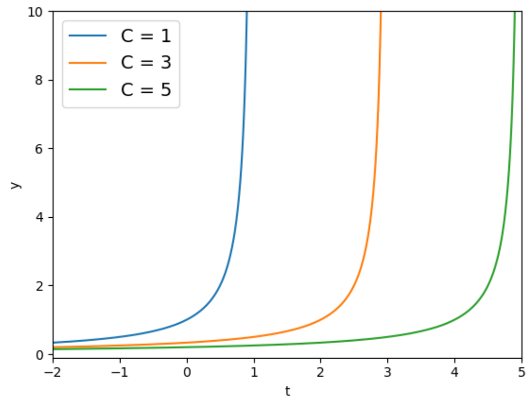

Figure 1: Solution of $y' - y^2 = 0$ for different choice of C

We finish our introduction by considering the so-called initial-value problem. We saw in Example 1.1 that there was a different solution to the differential equation for each value of the constant. What if we are interested in selecting a particular solution from this general family of solutions? Given the "data" $y(t_0) = y_0$ referred to as an **initial condition (IC)** we can solve for the constant C and thus obtain a particular solution from the one-parameter family of solutions.

Definition 1.3. An **initial-value problem (IVP)** is an ODE together with a set of specified values, called the initial conditions, of the unknown function at a given point in the domain of the solution. Generally speaking, an n^{th}-order ODE will require n independent conditions.

Example 1.2. Recall that the solution to the differential equation $y' = y^2$ is given by $y(t) = (C - t)^{-1}$. Now assume that we add the initial condition $y(0) = 1$. Then we have,

$$y(0) = \frac{1}{C} = 1 \iff C = 1.$$

Thus, we arrive at the **particular solution** $y(t) = (1 - t)^{-1}$ defined on the interval $-\infty < t < 1$. Thus, the interval of existence for the solution will depend on the initial condition (see Fig. 2).

Concerning IVPs we are primarily interested in the **existence** and **uniqueness** of solutions as well as techniques for solving (either exactly or approximately) said IVPs.

2 First-order differential equations:

In this section we will study the first-order differential equation given by

$$\frac{dy}{dt} = f(t, y), \tag{2.1}$$

and the corresponding initial-value problem given by

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{2.2}$$

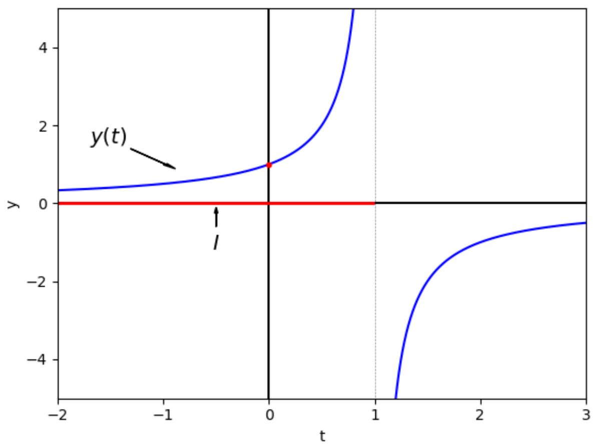

Figure 2: Solution to the initial-value problem $y' = y^2$, $y(0) = 1$. The red line is the interval on which the solution exists, i.e., the interval of existence.

2.1 Integrals as general and particular solutions:

First, consider the case where $f(t, y) = f(t)$, that is, f depends on t only. Then (2.1) takes the simple form

$$\frac{dy}{dt} = f(t). \tag{2.3}$$

By the fundamental theorem of calculus we have:

$$\int \frac{dy}{dt} dt = \int f(t)dt + C \iff y(t) = F(t) + C$$

where $F'(t) = f(t)$ and C is a constant of integration. Thus, we say

$$y(t) = F(t) + C$$

is the **general solution** to the differential equation (2.3). Next, consider the corresponding initial-value problem given by

$$\frac{dy}{dt} = f(t), \quad y(t_0) = y_0. \tag{2.4}$$

Integrating both sides of (2.4) gives

$$\int \frac{dy}{dt} dt = \int f(t)dt + C \iff y(t) = F(t) + C$$

where $F'(t) = f(t)$ and C is a constant of integration. To find C we use the initial condition. Namely,

$$y(t_0) = y_0 \implies F(t_0) + C = y_0 \iff C = y_0 - F(t_0)$$

and thus $y(t) = F(t) - F(t_0) + y_0 = y_0 + \int_{t_0}^{t} f(s)ds$. Hence, we say

$$y(t) = y_0 + \int_{t_0}^{t} f(s)ds \tag{2.5}$$

is the **particular solution** to the initial-value problem (2.4). So our first technique for solving a differential equation is integration. It's important to realize that simply integrating both sides of (2.3) is only a valid solution technique when f is a function of t only.

Example 2.1. Solve the initial-value problem

$$\frac{dy}{dt} = \cos^2 t, \qquad y(0) = \pi$$

To solve this initial-value problem we first note that the right hand side of the differential equation is a function of t only. Thus the technique we use is integration. To integrate $\cos^2 t$ we recall the trigonometric identity

$$\cos^2 t = \frac{1 + \cos 2t}{2}.$$

Thus,

$$y(t) = \pi + \int_0^t \frac{1 + \cos 2s}{2} ds = \pi + \left(\frac{1}{2}s + \frac{1}{4}\sin 2s\right)\Big|_0^t = \pi + \frac{1}{2}t + \frac{1}{4}\sin 2t.$$

Note that we can also check that our solution is correct by plugging it into the differential equation and making sure that the initial condition is satisfied. Hence,

$$y'(t) = \frac{1}{2} + \frac{2}{4}\cos 2t = \cos^2 t$$

and

$$y(0) = \pi + \frac{1}{2} \cdot 0 + \frac{1}{4}\sin 0 = \pi.$$

2.2 Velocity and acceleration:

Let's consider the one-dimensional motion of a particle.

$$y = y(t) \text{ position of the particle at time } t \quad [\text{Length}]$$
$$v = v(t) \text{ velocity of the particle at time } t \quad [\text{Length/Time}]$$
$$a = a(t) \text{ acceleration of the particle at time } t \quad [\text{Length/Time}^2]$$

It turns out that $v(t) = y'(t)$ and $a(t) = y''(t)$. That is, velocity equals the first derivative of position, and acceleration equals the second derivative of position. Next recall Newton's second law of motion, namely, force equals mass times acceleration.

$$F = ma$$

Using $a(t) = y''(t)$ we can derive a second-order differential equation whose unknown is the position function.

$$y'' = \frac{F(t)}{m}$$

We make the assumption that $F(t) = -mg$ where g is the gravitational constant in a uniform field. Thus,

$$y'' = -g,$$

and integrating twice gives

$$y(t) = -\frac{1}{2}gt^2 + C_1 t + C_2,$$

where C_1 and C_2 are the integration constants. Finally, given the initial position $y(0) = y_0$, and the initial velocity $v(0) = v_0$ we arrive at the position function for one-dimensional motion under the force of gravity in a uniform field,

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0. \tag{2.6}$$

2.3 Slope fields:

Let's turn our attention back to the general explicit first-order differential equation

$$\frac{dy}{dt} = f(t, y). \tag{2.7}$$

It turns out that there is no general procedure for solving (2.7). Integrating both sides of (2.7) gives

$$y(t) = \int f(t, y(t))dt + C,$$

which is not considered a solution since the unknown $y = y(t)$ appears on both sides of the equal sign. In essence we have taken a differential equation and transformed it into an integral equation.

We should not be discouraged by the fact that there is no general technique for solving (2.7). Actually the equation itself provides important geometric information which can be used to perform what is known as a 'qualitative analysis' of solutions to the differential equation. Recall that the derivative equals slope of the tangent line to a curve. Consider the point (t_0, y_0) in the ty-plane. Then $f(t_0, y_0)$ gives slope of the tangent line to a solution $y = y(t)$ of the given differential equation at (t_0, y_0). Thus, by plotting slope of the tangent line for many points in the ty-plane we can get a qualitative understanding of the overall behavior of solutions. Let's see an example.

2.4 Existence and uniqueness:

Before one spends much time attempting to solve a given differential equation, it is wise to know that solutions actually ***exist***. We may also want to know whether there is only one solution of the equation satisfying a given initial condition, that is, whether its solutions are ***unique***. The following theorem gives a criterion for the local existence and uniqueness of solutions to the initial-value problem (2.2). Later we will see a criterion for the global existence of solutions when the differential equation under consideration is assumed to be linear.

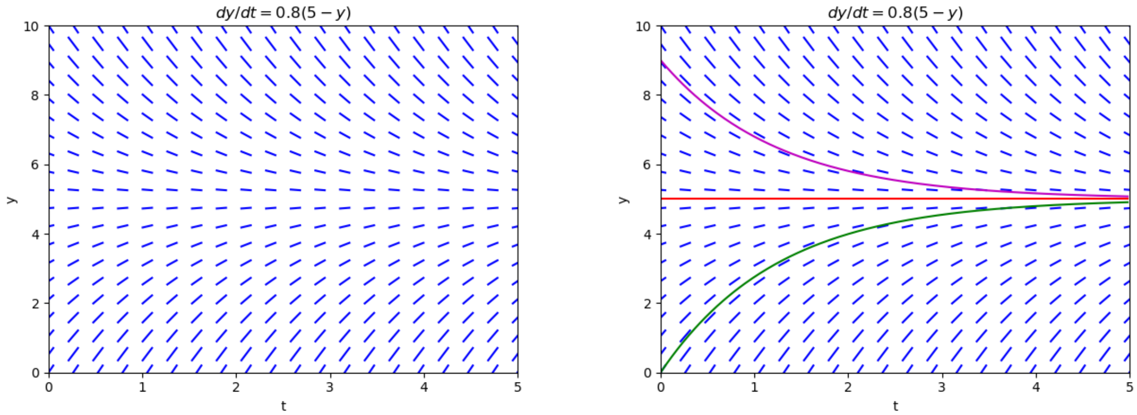

Figure 3: (Left) Slope field of the DE $y' = 0.8(5 - y)$. (Right) Slope field of the DE $y' = 0.8(5 - y)$ with solution curves corresponding to the ICs $y(0) > 5$ (purple), $y(0) = 5$ (red), and $y(0) < 5$ (green).

Theorem 2.1. Suppose that both the function $f(t, y)$ and its partial derivative $\frac{\partial f}{\partial y}(t, y)$ are continuous on some rectangle $\mathcal{R}$ in the ty-plane that contains the point (t_0, y_0) in its interior. Then, for some open interval I containing the point t_0, the initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

has one and only one solution that is defined on the interval I. (Note: The solution interval I may not be as 'wide' as the original rectangle $\mathcal{R}$ of continuity.)

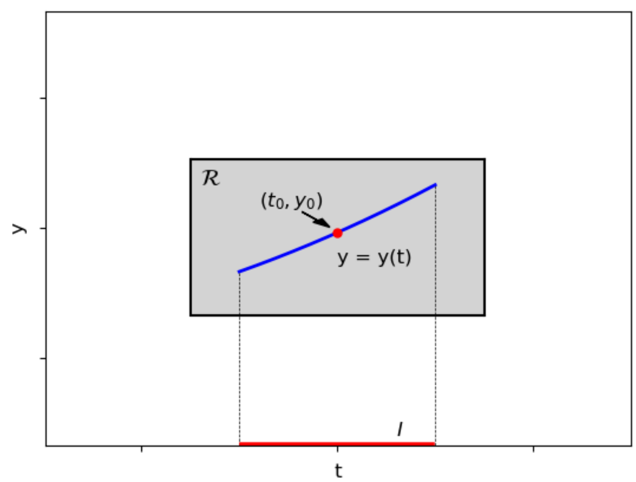

Figure 4: The rectangle $\mathcal{R}$ and t-interval I of Theorem 2.1, and the solution curve $y = y(t)$ through the point (t_0, y_0).

To prove Theorem 2.1 we note that $y = y(t)$ solves the initial-value problem $y' = f(t, y)$, $y(t_0) = y_0$ if and only if it solves the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds. \tag{2.8}$$

Then we employ the **method of successive approximations** to 2.8, which was developed by the French mathematician Emile Picard (1856-1941). The idea is to construct a sequence of approximate solutions as follows

$$\begin{aligned}
y_1(t) &= y_0 + \int_{t_0}^{t} f(s, y_0)ds, \\
y_2(t) &= y_0 + \int_{t_0}^{t} f(s, y_1(s))ds, \\
y_n(t) &= y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))ds.
\end{aligned}$$

Then we show that the $y_n(t) \to y(t)$ uniformly, and that $y(t)$ satisfies the initial-value problem. Finally, to prove uniqueness we assume that another solution to the initial-value problem exists and derive a contradiction.

2.5 Separable equations:

Next let's consider the case where $f(t, y) = h(t)g(y)$. A differential equation that can be written as

$$\frac{dy}{dt} = h(t)g(y) \tag{2.9}$$

is called **separable**. To solve Eq (2.9) consider the following steps:

$$\text{Step 1: (rewrite)} \qquad \frac{1}{g(y)}dy = h(t)dt.$$

$$\text{Step 2: (integrate)} \qquad \int \frac{1}{g(y)}dy = \int h(t)dt.$$

$$\text{step 3: (solve)} \qquad G(y) = H(t) + C.$$

Example 2.2. Solve the differential equation $y' = 2y$. We begin by noting that the differential equation is separable since the right hand side depends on y only. Thus we rewrite as

$$\frac{1}{y}dy = 2dt.$$

Next, integrate both sides

$$\int \frac{1}{y}dy = 2\int dt.$$

Finally, try to solve for y

$$\ln|y| = 2t + C \iff e^{\ln|y|} = e^{2t+C} \iff y(t) = Ae^{2t}, \quad A = e^C \neq 0.$$

Thus we arrived at the one-parameter family of solutions $y(t) = Ae^{2t}$.

An important question to ask now is did we find all of the solutions? Suppose we let $A = 0$. Then $y(t) \equiv 0$ which is indeed a solution. Since $A = e^C \neq 0$ the function $y(t) \equiv 0$ does not live in our one-parameter family of solutions despite solving the differential equation. So why did we miss this solution? Essentially, when dividing by y we implicitly made the assumption $y \neq 0$ and thus our one-parameter family does not consider $y(t) \equiv 0$ as a possible solution. We call $y(t) \equiv 0$ a **singular solution** of the differential equation.

Example 2.3. Solve the initial-value problem $\frac{dy}{dt} = \frac{4-2t}{3y^2-5}$, $y(1) = 3$. We begin by noting that the differential equation is separable. Thus we rewrite as

$$3y^2 - 5dy = 4 - 2tdt.$$

Next, integrate both sides

$$\int 3y^2 - 5dy = \int 4 - 2tdt \iff y^3 - 5y = 4t - t^2 + C.$$

Note that we cannot solve for y in terms of t explicitly. We can still use the initial condition to find C. Thus,

$$3^3 - 5(3) = 4(1) - 1^2 + C \iff C = 9,$$

and the implicitly defined solution of the initial value problem is given by $y^3 - 5y = 4t - t^2 + 9$.

Notice that in Example 2.3 we could not find $y = y(t)$ explicitly. The equation $K(t, y) = 0$ is commonly called an **implicit solution** of a differential equation if it is satisfied (on some interval) by some solution $y = y(t)$ of the differential equation. But note that a particular solution $y = y(t)$ of $K(t, y) = 0$ may or may not satisfy a given initial condition.

Remark 1: Notice that in Example 2.3 $\frac{dy}{dt}$ is undefined when $y = \pm\sqrt{\frac{5}{3}}$. This separates the ty-plane into three distinct regions. Each particular solution is confined the one of these three regions.

Remark 2: Consider the general solution in Example 2.3 given by $F(t, y) = y^3 - 5y - (4t - t^2) = C$. Thus the one-parameter family of solutions is equivalent to the level curves given by $F(t, y) = C$.

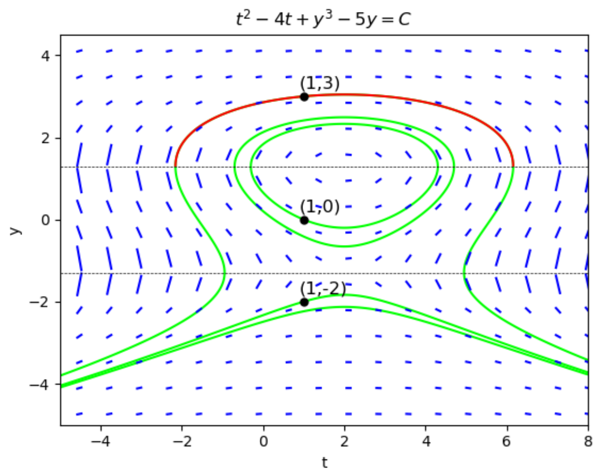

Figure 5: Light green: Level curves of $t^2 - 4t + y^3 - 5y = C$ where $C = -3, -1$, and 9. Red: Particular solution of the initial-value problem in Example 2.3. Black dashed: Lines $y = \pm\sqrt{5/3}$. Blue: Slope field.

Remark 3: You should not assume that every possible algebraic solution $y = y(t)$ of an implicit solution satisfies the same differential equation. For instance $t^2+y^2-4 = 0$ is and implicit solution of the differential equation $t + yy' = 0$. Multiplication by $(y - 2t)$ gives the new implicit solution $(y - 2t)(t^2 + y^2 - 4) = 0$, but it is easy to check that $y(t) = 2t$ does *not* satisfy this differential equation.

Remark 4: Similarly, solutions of a given differential equation can be either gained or lost when it is multiplied or divided by an algebraic factor. Consider $(y-2t)yy' = -t(y-2t)$ having the obvious solution $y(t) = 2t$. If we divide both sides by $(y - 2t)$ we get the differential equation from Remark 3, namely, $t + yy' = 0$ of which $y(t) = 2t$ is *not* a solution. Thus we 'lose' the solution $y(t) = 2t$ to the differential equation $(y - 2t)yy' = -t(y - 2t)$ upon its division by the factor $(y - 2t)$.

Remark 5: A solution of a differential equation that contains an 'arbitrary' constant C is called a **general solution** of the differential equation. Any specific choice of the constant C gives a **particular solution** of the differential equation. We will see that every particular solution of a *linear* differential equation is contained in its general solution. By contrast, it is common for a nonlinear differential equation to have both a general solution involving an arbitrary constant C and one or several particular solutions that cannot be obtained by selecting a value for C. These exceptional solutions are frequently called **singular solutions**.

2.5.1 Applications of separable equations:

Some applications of separable ODEs are now presented. In each case we will express a proposed physical relationship between variables as a separable differential equation.

Exponential growth and decay:

The time rate of change of a quantity is proportional to its current size

$$\frac{dy}{dt} = \pm ky, \qquad k > 0 \tag{2.10}$$

Here we let $y = y(t)$ be the amount of some quantity, and k be the proportionality constant. Equation (2.10) serves as a mathematical model for a wide range of natural phenomena. For $+k$ applications include population growth with constant birth and death rates, and continuously compounded interest. In this case we call k the relative growth rate. For $-k$ applications include radiocarbon dating, and drug elimination. In this case we call k the decay constant.

It is easy to see that Eq. (2.10) is a separable ODE. If we let $y(0) = y_0$ be the initial condition then the IVP has solution

$$y(t) = y_0 e^{\pm kt}, \tag{2.11}$$

where y_0 is the initial amount of the quantity.

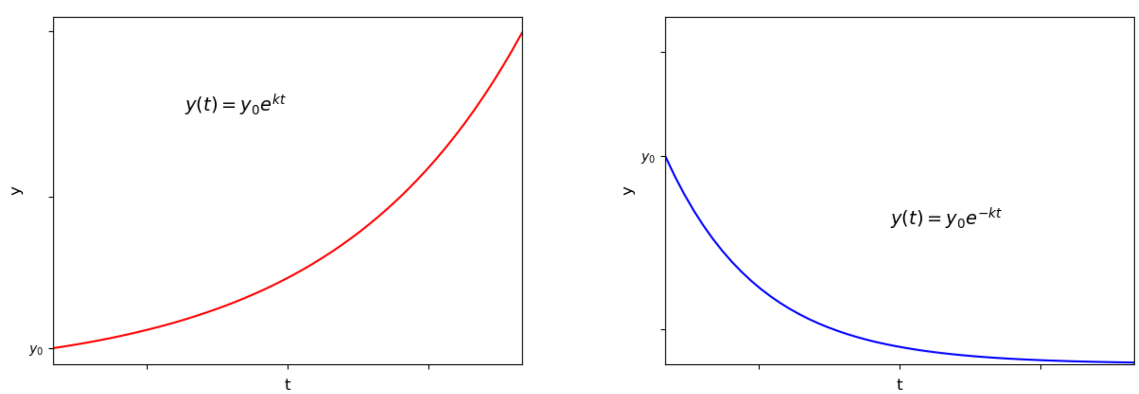

Figure 6: (Left) Exponential growth with $k = 0.1$ and $y_0 = 8$. (Right) Exponential decay with $k = -0.2$ and $y_0 = 60$.

Cooling and heating:

Newton's law of cooling: *The time rate of change in the temperature of an object is proportional to the difference between the object's current temperature and the temperature of the surrounding.*

$$\frac{dT}{dt} = -k(T - R), \qquad k > 0, \tag{2.12}$$

Here we let $T = T(t)$ be the object's temperature. Also, R is the surrounding temperature, and k is the proportionality constant. One way to find k is by lab experiments.

It is easy to see that Eq. (2.12) is a separable ODE. If we let $T(0) = T_0$ be the initial condition then the IVP has solution

$$T(t) = R + (T_0 - R)e^{-kt}, \tag{2.13}$$

where T_0 is the object's initial temperature.

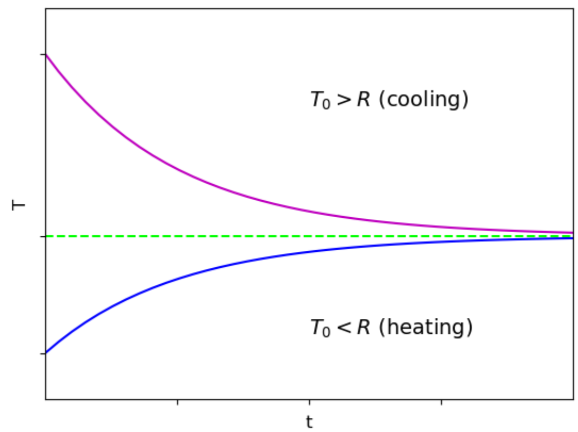

Figure 7: Solutions to (2.12) for $R = 50°$ and $k = 0.1$. (Light green) Equilibrium solution, i.e, $T_0 = R$. (Magenta) Cooling solution, i.e., $T_0 > R$. (Blue) Heating solution, i.e., $T_0 < R$.

Torricelli's law:
Suppose that a water tank has a hole of radius a at its bottom, from which water is leaking. Denote by $y = y(t)$ the depth of water in the tank at time t, and by $V = V(t)$ the volume of water in the tank then. By conservation of energy the velocity of water leaving the tank through the hole is given by

$$v = \sqrt{2gy},$$

where g is the gravitational constant. So,

$$\begin{aligned}\frac{dV}{dt} &= -av \\ &= -a\sqrt{2gy} \\ &= -k\sqrt{y},\end{aligned}$$

where $k = a\sqrt{2g}$. Thus we arrive at the differential equation

$$\frac{dV}{dt} = -k\sqrt{y} \tag{2.14}$$

which is Torricelli's law for a draining tank. Next, let $A(y)$ denote the horizontal cross-sectional area of the tank at height y. Then by applying the method of cross-sections from calculus we arrive at the following volume formula:

$$V(y) = \int_0^y A(\bar{y})d\bar{y}.$$

Finally, applying the chain rule and fundamental theorem of calculus gives:

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}.$$

Hence, we arrive at the alternate form of Torricelli's law, namely,

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \tag{2.15}$$

Remark 6: When solving application based problems you may need to write a relevant differential equation, use geometric intuition, or find parameters using given data. One must understand how to use the data provided within the problem to come up with a solution.

2.6 Linear first-order equations:

A **linear first-order equation** is a differential equation of the form

$$A(t)\frac{dy}{dt} + B(t)y = C(t). \tag{2.16}$$

By dividing both sides of Eq. (2.16) by $A(t)$ we arrive at the alternate form of a linear first-order equation, namely,

$$\frac{dy}{dt} + P(t)y = Q(t), \tag{2.17}$$

where $P(t) = B(t)/A(t)$ and $Q(t) = C(t)/A(t)$. Any first-order equation not of the forms (2.16) or (2.17) is called nonlinear.

To solve Eq. (2.17) we introduce the notion of an integrating factor. An **integrating factor** for a differential equation is a function $\rho = \rho(t, y)$ such that multiplication of each side of the differential equation by ρ yields an equation which is recognizable as a derivative. In some cases the integrating factor may be a function of t, or y, only. Let's try to find an integrating factor for Eq. (2.17).

Begin by multiplying Eq. (2.17) by some unknown function $\rho = \rho(t)$.

$$\rho y' + \rho P(t)y = \rho Q(t)$$

Try to rewrite the left-hand side of the above equation as a derivative.

$$\begin{aligned} \rho' y + \rho y' &= \frac{d}{dt}(\rho y) \\ &= \rho y' + \rho P(t)y \end{aligned}$$

Canceling like terms we have a separable differential equation, namely,

$$\rho' = \rho P(t)$$

whose solution is given by $\rho(t) = e^{\int P(t)dt}$.

Hence we have found an integrating factor for the linear-first order differential equation (2.17).

$$\rho(t) = e^{\int P(t)dt} \tag{2.18}$$

Now using the integrating factor (2.18) we try to solve Eq. (2.17).

$$\begin{aligned}\frac{d}{dt}(\rho(t)y(t)) = \rho(t)Q(t) &\iff \rho(t)y(t) = \int \rho(t)Q(t)dt + C \\ &\iff y(t) = \frac{1}{\rho(t)}\Big(\int \rho(t)Q(t)dt + C\Big)\end{aligned}$$

Also note that since $\rho(t) = e^{\int P(t)dt} \neq 0$ for all t division by ρ is a well-defined operation. Thus,

$$y(t) = \frac{1}{\rho(t)}\Big(\int \rho(t)Q(t)dt + C\Big) \tag{2.19}$$

is the general solution to the first-order linear differential equation.

Theorem 2.2. If the functions $P(t)$ and $Q(t)$ are continuous on the open interval I containing the point t_0, then the initial-value problem

$$\frac{dy}{dt} + P(t)y = Q(t), \quad y(t_0) = y_0 \tag{2.20}$$

has a unique solution $y = y(t)$ on I, given by the formula (2.19) with an appropriate value of C.

Remark 7: Theorem 2.2 gives a solution on the *entire* interval I where P, and Q are continuous for a *linear* differential equation. This is in contrast to Theorem 2.1 which guarantees only a solution on a possible smaller interval.

Remark 8: Theorem 2.2 tells us that every solutions is contained in the general solution (2.19). Thus, the *linear* first-order differential equation has no singular solutions.

To solve Eq. (2.17) consider the following steps:

Step 1: Calculate the integrating factor $\rightarrow \rho(t) = e^{\int P(t)dt}$.

Step 2: Multiply both sides of Eq. (2.17) by $\rho(t)$.

Step 3: Recognize the left-hand side as the derivative $\rightarrow \frac{d}{dt}(\rho(t)y(t)) = \rho(t)Q(t)$.

Step 4: Integrate both sides and solve for $y = y(t) \rightarrow \rho(t)y(t) = \int \rho(t)Q(t)dt + C$.

Example 2.4. Solve the initial-value problem $3ty' + y = 12t$, $y(1) = 2$. First we rewrite in the standard form $y' + \big(\frac{1}{3t}\big)y = 4$. Thus, $\rho(t) = e^{\int \frac{1}{3t}dt} = e^{\frac{1}{3}\ln t} = t^{1/3}$. Applying step 2 gives

$$t^{1/3}y' + \frac{1}{3}t^{-1}t^{1/3}y = 4t^{1/3} \iff t^{1/3}y' + \frac{1}{3}t^{-2/3}y = 4t^{1/3} \iff \frac{d}{dt}\big(t^{1/3}y\big) = 4t^{1/3}.$$

Next, integration yields $t^{1/3}y = 4\int t^{1/3}dt = 3t^{4/3} + C$. Finally we arrive at the general solution, namely,

$$y(t) = 3t + Ct^{-1/3}.$$

Next, using the initial condition gives $y(1) = 3 - C = 2 \iff C = -1$. Hence, the particular solution is given by

$$y(t) = 3t - t^{-1/3}.$$

2.6.1 Applications of linear first-order equations:

Mixture

Consider a tank containing a solution - mixture of solute and solvent - such as salt dissolved in water. There is both inflow and outflow, and we want to compute the *amount* $x = x(t)$ of solute in the tank at time t, given the amount $x(0) = x_0$ at time $t = 0$. Suppose that solution with concentration c_i (g/L) flows into the tank at the constant rate r_i (L/s), and that the solution in the tank - kept thoroughly mixed by stirring - flows out at the constant rate r_o (L/s). To set up a differential equation for $x = x(t)$, we estimate the change Δx in x during the time interval $[t, t + \Delta t]$ as follows:

$$\begin{aligned}\Delta x &= \{\text{amount in}\} - \{\text{amount out}\} \\ &\approx r_i c_i \Delta t - r_o c_o \Delta t\end{aligned}$$

Thus, $\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o$. Finally, noting that $c_o = x/V$, and letting $\Delta t \to 0$ gives the differential equation

$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i \tag{2.21}$$

for the amount of solute in the tank. Note that Eq. (2.21) is linear first-order, and in general we can write $V(t) = V_0 + (r_i - r_o)t$.

Example 2.5. A tank contains 1000 liters (L) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s, and the mixture - kept uniform by stirring - is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

We begin by using the data to set up the relevant differential equation. Since pure water is pumped into the tank we must have $c_i = 0$. Also, since $r_i = r_o$ the volume is constant and equals $V = 1000$ L. Also, the initial amount of salt $x(0) = 100$ kg is given. Thus we have the equation

$$\frac{dx}{dt} = -\frac{x}{V}r_o,$$

where $V = 1000$ L and $r_o = 5$ L/s. Thus, the solution is given by $x(t) = x_0 e^{-\frac{1}{200}}t$, where $x_0 = 100$ kg. Finally, to answer the equation we solve the equation

$$100e^{-\frac{1}{200}t} = 10$$

for t. Thus there is 10 kg of salt in the tank after roughly 7 minutes and 40 seconds.

2.7 Substitution methods and exact equations:

In this section we first look at some differential equations that can be solve by first making a substitution of the dependent variable.

Form 1: $y' = F(at + by + c)$
Substitution: $v = at + by + c$ and $v' = a + by'$

Form 2: $y' = F\big(\frac{y}{t}\big)$
Substitution: $y = vt$ and $y' = tv' + v$
Homogeneous type: $At^m y^n y' + Bt^p y^q = Ct^r y^s$, where $m + n = p + q = r + s = K$

Form 3: $y' + P(t)y = Q(t)y^n$
Substitution $v = y^{1-n}$ and $v' = (1-n)y^{-n}y'$
Bernoulli type: $n = 0$ is linear first-order, $n = 1$ is separable, $n \neq 0, 1$ make the substitution

Form 4: $M(x,y)dx + N(x,y)dy = 0$
Substitution: $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$
Exact: if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Form 5: $F(t, y', y'') = 0$
Substitution: $p = y'$ and $p' = y''$
Reducible: dependent variable missing

Form 6: $F(y, y', y'') = 0$
Substitution: $p = y'$ and $p\frac{dp}{dy} = y''$
Reducible: independent variable missing

Example 2.6. Solve the differential equation $(t^2 - y^2)y' = 2ty$. Begin by multiplying the equation by $1/t^2$. This gives the equivalent DE $(1 - (y/t)^2)y' = 2(y/t)$. This has form 2 above thus we make the substitution $v = y/t \iff y = vt$ and $y' = tv' + v$. Thus, $(1 - v^2)(tv' + v) = 2v$ which we rewrite in the separated form:

$$\int \frac{v^2 - 1}{v(v^2 + 1)} dv = -\int \frac{1}{t} dt$$

To integrate the left-hand side we must use partial fractions.

$$\frac{v^2 - 1}{v(v^2 + 1)} = \frac{A}{v} + \frac{Bv + C}{v^2 + 1} \iff \frac{v^2 - 1}{v(v^2 + 1)} = \frac{(A + B)v^2 + Cv + A}{v(v^2 + 1)}$$

Equating the coefficients gives $A + B = 1$, $Cv = 0$ and $A = -1$. Thus we have the solution $A = -1$, $B = 2$ and $C = 0$. Thus we integrate:

$$\int \Big(\frac{-1}{v} + \frac{2v}{v^2 + 1}\Big) dv = -\int \frac{1}{t} dt \iff -\ln|v| + \ln(v^2 + 1) = -\ln|t| + C$$

Finally, simple algebra yields the implicit solution $y = C(y^2 + t^2)$.

Example 2.7. Check that the differential equation $\big(x^3 + \frac{y}{x}\big)dx + \big(y^2 + \ln x\big)dy = 0$ is exact. Then solve it.

To check that the equation is exact we calculate:

$$\frac{\partial M}{\partial y} = \frac{1}{x}$$
$$\frac{\partial N}{\partial x} = \frac{1}{x}$$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation is exact. To solve it we first evaluate

$$\int M(x,y)dx = \int \left(x^3 + yx^{-1}\right) = \frac{1}{4}x^4 + y\ln x + g(y)$$

Next we take the partial derivative of the above expression with respect to y.

$$\frac{\partial}{\partial y}\left(\frac{1}{4}x^4 + y\ln x + g(y)\right) = \ln x + g'(y)$$

Finally, we equate the above expression with $N(x,y)$ arriving at the expression for $g'(y)$.

$$g'(y) = y^2 \iff g(y) = \frac{1}{3}y^3 + C$$

Hence the solution is given implicitly by $\frac{1}{4}x^4 + \frac{1}{3}x^3 + y\ln x = C$.

2.8 Equilibrium solutions and stability:

In this section we perform a **qualitative analysis** of solutions to a differential equation. This is a powerful geometric approach to the study of differential equations. We begin with some definitions.

An **autonomous** differential equation is an equation of the form:

$$\frac{dy}{dt} = f(y), \tag{2.22}$$

i.e., the function f depends on the dependent variable only.

Solutions of the equation $f(y) = 0$ play an important role and are called **critical points** of the autonomous differential equation. If $y = c$ is a critical point of (2.22), then the differential equation has the constant solution $y(t) \equiv c$. A constant solution of a differential equation is called an **equilibrium solution**.

A critical point $y = c$ of an autonomous differential equation is said to be *stable* provided that, if the initial value y_0 is sufficiently close to c, then $y(t)$ remains close to c for all $t > 0$. More precisely, the critical point c is **stable** if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|y_0 - c| < \delta \qquad \text{implies that} \qquad |y(t) - c| < \epsilon$$

for all $t > 0$. The critical point $y = c$ is **unstable** if it is not stable.
Let's now perform a qualitative analysis of an autonomous differential equation.

Example 2.8. Sketch the one-parameter family of solutions to the differential equation $y' = y^2 - y - 2$ by performing a qualitative analysis. Draw a phase diagram. Classify the equilibrium solutions as stable or unstable.

(Step 1:) Find all equilibrium solutions.

$$y' = 0 \iff y^2 - y - 2 = 0 \iff (y-2)(y+1) = 0$$

Thus, the equilibrium solutions are given by $y(t) \equiv 2$ and $y(t) \equiv -1$.

(Step 2:) Find where solutions are increasing and decreasing. By continuity the sign of the derivative does not change in each interval $(-\infty, -1)$, $(-1, 2)$, and $(2, \infty)$.

(i.) $y > 2$: $y' = (y-2)(y+1) > 0 \implies y$ increasing

(ii.) $-1 < y < 2$: $y' = (y-2)(y+1) < 0 \implies y$ decreasing

(iii.) $y < -1$: $y' = (y-2)(y+1) > 0 \implies y$ increasing

(Step 3:) Check concavity of solutions. We begin by locating possible inflection points by solving the equation $y'' = 0$. By the chain rule:

$$y'' = \frac{d}{dt}f(y) = \frac{df}{dy}\frac{dy}{dt} = (2y-1)y' = 0 \iff y = 1/2 \quad \text{or} \quad y' = 0$$

(i.) $y > 2$: $y'' = (2y-1)y' > 0 \implies y$ concave up

(ii.) $1/2 < y < 2$: $y'' = (2y-1)y' < 0 \implies y$ concave down

(iii.) $-1 < y < 1/2$: $y'' = (2y-1)y' > 0 \implies y$ concave up

(iv.) $y < -1$: $y'' = (2y-1)y' < 0 \implies y$ concave down

Using the above qualitative information we can plot some representative solutions to the differential equation in the ty-plane.

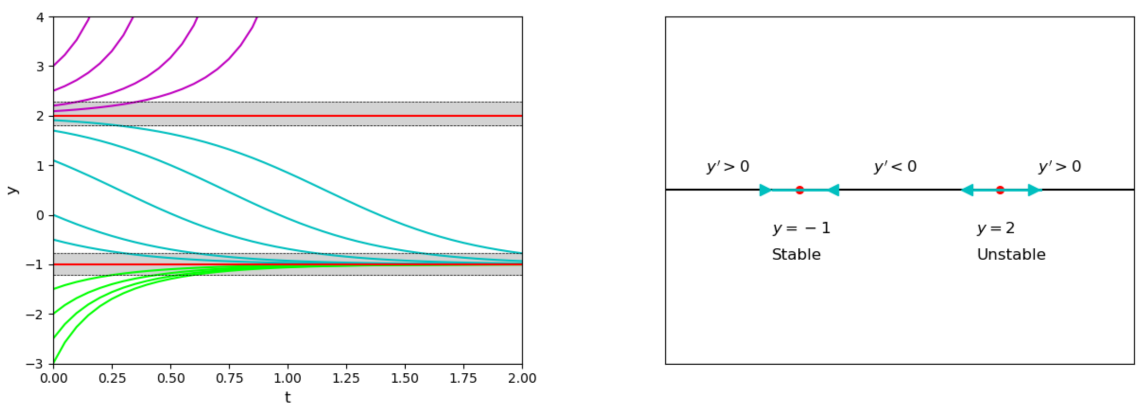

Figure 8: (Left) One parameter family of solutions to the differential equation $y' = y^2 - y - 2$ found by qualitative analysis. (Right) Phase diagram for the differential equation $y' = y^2 - y - 2$. We see that $y = -1$ is a stable equilibrium solution while $y = 2$ is an unstable equilibrium solution.

Earlier we discussed the exponential growth and decay equation (2.10). For population dynamics Eq. (2.10) assumes constant birth and death rates. A more accurate population model may assume that the birth rate decreases as the population increases. In this case we have

$$\frac{dP}{dt} = kP(M - P), \qquad P(0) = P_0 \tag{2.23}$$

where $P = P(t)$ is the population at time t, and M is the *carrying capacity* (maximum sustainable population). We call (2.23) the **logistic equation**. A qualitative analysis of Eq. (2.23) shows that for $0 < P_0 < M$ we have $P' > 0$, and for $P_0 > M$ we have $P' < 0$. Thus, the logistic equation has a limiting population, i.e., $P(t) \to M$ as $t \to \infty$. This is in contrast to the exponential growth equation and can be attributable to finite resources needed for survival. If instead we write Eq. (2.23) as $P' = kP(P - M)$ we now call M a *threshold population*, with the way the population behaves in the future depending critically on whether the initial population P_0 is less than or greater than M.

A biological or physical system that is modeled by a differential equation may depend crucially on the numerical values of certain coefficients or parameter that appear in the equation. For example

$$\frac{dy}{dt} = ky(M - y) - h, \tag{2.24}$$

is the logistic equation with a *harvesting* parameter h. Think of a fish population in a lake from which h fish per year are removed by fishing. Note that we can rewrite Eq. (2.24) as $y' = ay^2 + by - h$, where $a = -k$, and $b = kM$. Thus the number of equilibrium solutions depends on the parameter h. By the quadratic formula

$$y' = 0 \iff y = \frac{1}{2}\left(M \pm \sqrt{M^2 - 4h/k}\right).$$

There are three cases to consider: (i.) $h < kM^2/4$ (two equilibrium solutions), (ii.) $h = kM^2/4$ (one equilibrium solution), and (iii.) $h > kM^2/4$ (zero equilibrium solutions). We call $h = kM^2/4$ a **bifurcation point** since at this point there is a qualitative change in solutions to Eq. (2.24). A common way to understand the 'bifurcation' in solutions is to plot the **bifurcation diagram** consisting of all points (h, c), where c is a critical point of the equation $y' = ky(M - y) - h$.

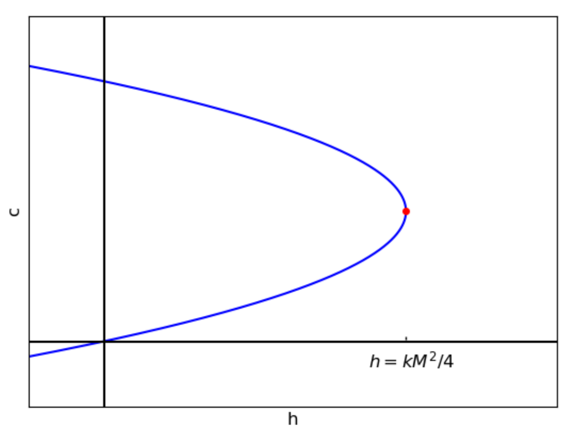

Figure 9: Bifurcation diagram for the logistic equation with harvesting.

For another example of autonomous differential equations recall Newton's second law of motion, namely, force equals mass times acceleration. Under the assumption that constant gravitational acceleration is the only force acting on a projectile we derived Eq. (2.6) for one-dimensional motion. To add more accuracy to our predictions let's also consider the force due to air resistance. Thus, the relevant equation is

$$m\frac{dv}{dt} = F_G + F_R,$$

where m is mass, $v = v(t)$ is velocity, F_G is force due to gravity, and F_R force due to air resistance. Note that F_R always acts in the direction opposite to the projectile's motion.

$\mathbf{F_R}$ proportional to velocity:

$$m\frac{dv}{dt} = -mg - kv \tag{2.25}$$

$\mathbf{F_R}$ proportional to velocity squared:

$$m\frac{dv}{dt} = -mg - kv|v| \tag{2.26}$$

Further, unless a projectile in vertical motion remains in the immediate vicinity of the earth's surface, the gravitational acceleration acting on it is not constant. According to Newton's law of gravitation, the gravitational force of attraction between two point masses M and m located at a distance r apart is given by

$$F_G = \frac{GMm}{r^2}, \tag{2.27}$$

where G is a certain empirical constant ($G \approx 6.6726 \times 10^{-11} N\cdot$(m/kg)^2 in mks units).

Example 2.9. What initial velocity v_0 is necessary for a projectile of mass m to escape from the earth? He we let $r = r(t)$ be the projectile's distance from the earth's center at time t. Then we have the equation

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2},$$

which is a reducible second-order differential equation. By the chain rule $dv/dt = v(dv/dr)$. This gives

$$vdv = -\frac{GM}{r^2}dr \iff \frac{1}{2}v^2 = \frac{GM}{r} + C$$

Now $v = v_0$ and $r = R$ when $t = 0$, so $C = v_0^2/2 - GM/R$. Here R is the earth's approximate radius. Thus, we have the implicit solution

$$v^2 = v_0^2 + 2GM\Big(\frac{1}{r} - \frac{1}{R}\Big).$$

It follows that

$$v^2 > v_0^2 - \frac{2GM}{R},$$

and so v will remain positive so long as $v_0^2 \geq 2GM/R$. Therefore the **escape velocity** from the earth is given by

$$v_0 = \sqrt{\frac{2GM}{R}}. \tag{2.28}$$

2.9 Euler's method:

Euler's method is a numerical algorithm for approximating the solution of a first-order differential equation. Recall the initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{2.29}$$

We have seen that while in special cases we can find the unknown function there is no general method for solving Eq. (2.29). For example, the differential equation

$$\frac{dy}{dt} = e^{-t^2},$$

has no solution that can be expressed in terms of elementary functions. Despite the lack of exact methods for solving Eq. (2.29) numerical algorithms exist that can compute approximate solutions. One such algorithm is Euler's method. It uses the fact that

$$y(t) \sim y(t_0) + y'(t_0)h \quad \text{as} \quad h \to 0, \tag{2.30}$$

to approximate the solution. Thus, since $y' = f(t, y)$ we can rewrite Eq. (2.30) as

$$y(t) = y_0 + f(t_0, y_0)h + o(h). \tag{2.31}$$

Since (t_0, y_0) are given we have a linear approximation of $y = y(t)$ for h sufficiently small. Continuing in this fashion we can construct an approximate solution over a range of t values, say, $t \in [t_0, t_{\text{max}}]$.

The algorithm goes as follows. Given the initial data (t_0, y_0) choose a step size h. Then make the following calculations:

$$\begin{array}{ll} t_1 = t_0 + h & \qquad y_1 = y_0 + f(y_0, t_0)h \\ t_2 = t_1 + h & \qquad y_2 = y_1 + f(y_1, t_1)h \\ \quad\vdots & \qquad\quad \vdots \\ t_n = t_{n-1} + h & \qquad y_n = y_{n-1} + f(y_{n-1}, t_{n-1})h \end{array}$$

Hence, the calculations above give a sequence of points $\{(t_i, y_i)\}_{i=0}^{n}$ which correspond to the approximate solution of Eq. (2.29). Since the algorithm involves lots of calculations it is best handled by scientific computing.

Remark 9: Generally the approximation gets better as the step-size h gets smaller.

Remark 10: The cumulative error in the approximation grows as we get farther from the initial data.

Remark 11: Euler's method is $O(h)$. This means that the *global truncation error* is (approximately) proportional to h.

Remark 12: Another algorithm called fourth-order Runge-Kutta has global truncation error $O(h^4)$. This is the algorithm used in practice due to its increased accuracy.

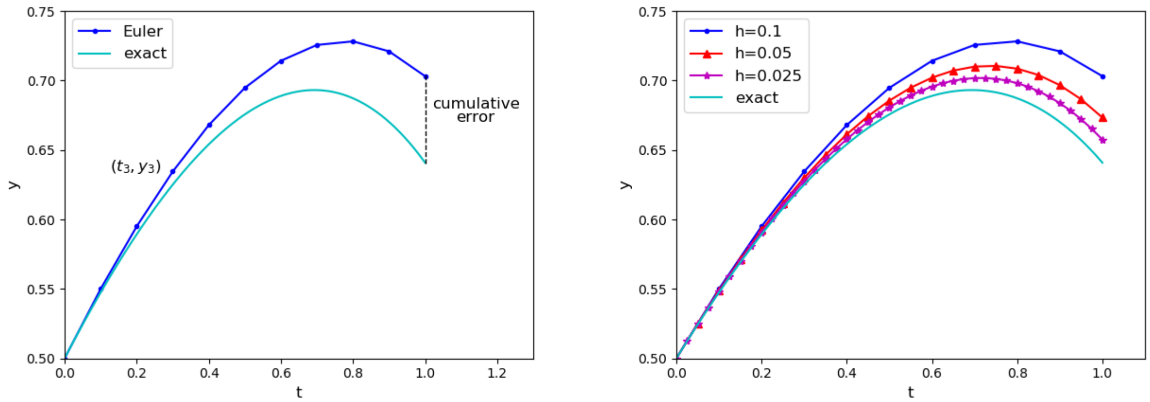

Figure 10: Approximate solution of the initial-value problem $y' = y - t$, $y(0) = 1$ by Euler's method along with the exact solution $y = t + 1 - e^t/2$.

3 Linear equations of higher order:

Recall that the general n-th order differential equation is given by

$$F(t, y, y', y'', \ldots, y^{(n-1)}, y^{(n)}) = 0, \tag{3.1}$$

where $y = y(t)$ is the unknown defined on an open interval I. In this section we are interested primarily in n-th order **linear** differential equations.

3.1 Second-order linear equations:

We begin by analyzing the linear second-order differential equation as the general theory parallels this case. A linear second-order differential equations has the form

$$A(t)y'' + B(t)y' + C(t)y = F(t), \tag{3.2}$$

where we will assume that the functions A, B, C and F are continuous on an open interval I. If we further assume that $A(t) \neq 0$ on I then we can write Eq. (3.2) in the standard form

$$y'' + p(t)y' + q(t)y = f(t), \tag{3.3}$$

where $p(t) = B(t)/A(t)$, $q(t) = C(t)/A(t)$ and $f(t) = F(t)/A(t)$. Also, if $f(t) \equiv 0$ then we call

$$y'' + p(t)y' + q(t)y = 0, \tag{3.4}$$

the associated **homogeneous** linear equation.

Theorem 3.1. Principle of Superposition
Let y_1 and y_2 be two solutions of the homogeneous linear equation in (3.4) on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution of Eq. (3.4).

Theorem 3.2. Existence and uniqueness for linear equations
Suppose that the functions p, q, and f are continuous on the open interval I containing the point t_0. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(t)y' + q(t)y = f(t), \tag{3.5}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(t_0) = b_0, \quad y'(t_0) = b_1. \tag{3.6}$$

Remark 1: Equation (3.5) and the conditions (3.6) constitute a second-order linear **initial-value problem**. Theorem 3.1 tells us that any such initial-value problem has a unique solution on the *whole* interval I where the coefficient functions in (3.5) are continuous. This is in contrast to the existence-uniqueness theorem for general first-order equations which only guarantees the existence and uniqueness of a solution in a neighborhood of the initial condition.

Remark 2: Unlike the first-order equation in which only a single solution curve $y = y(t)$ passes through a given initial point (t_0, b_0) the second-order equation has many solution curves passing through a given initial point, one for which (real number) value of the initial slope $y'(t_0) = b_1$.

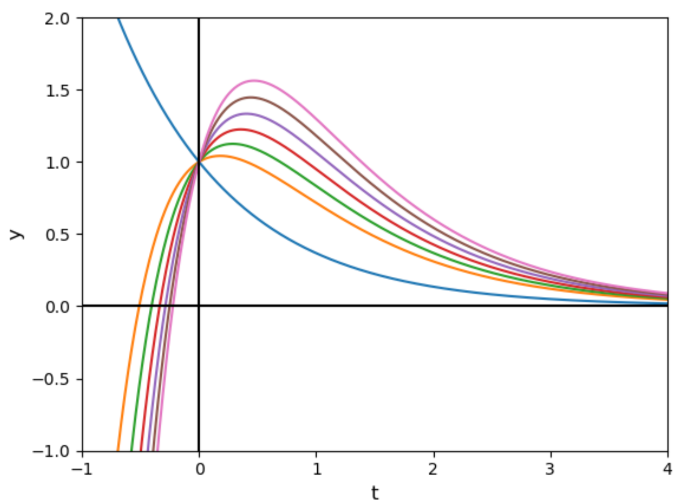

Figure 11: Solutions to the initial-value problem $y'' + 3y' + 2y = 0$,
$y(0) = 1$, and $y'(0) = b$. In this example $b \in \{-1, 0.5, 1, 1.5, 2, 2.5, 3\}$.

Definition 3.1. Linear independence of two functions
Two functions defined in a open interval I are said to be **linearly independent** on I provided that neither is a constant multiple of the other.

A function that is not linearly independent is said to be **linearly dependent**. Two functions are linearly dependent in a open interval I if and only if there exist constants c_1 and c_2 not both zero such that

$$c_1 f_1(t) + c_2 f_2(t) = 0,$$

for all $t \in I$. Note that if the ratio of two functions is a constant then they're linearly dependent. Another test for the linear independence, or dependence, of two functions is the **Wronskian**. The Wronskian of two functions is defined as

$$W[f_1, f_2](t) := \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1', \tag{3.7}$$

which is simply the determinant of a 2×2 matrix.

Theorem 3.3. Wronskian of solutions
Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation (Eq. (3.4))

$$y'' + p(t)y' + q(t)y = 0,$$

on an open interval I on which p and q are continuous.

(a) If y_1 and y_2 are linearly dependent, then $W[y_1, y_2](t) \equiv 0$ on I.

(b) If y_1 and y_2 are linearly independent, then $W[y_1, y_2](t) \neq 0$ at each point of I.

Thus the Wronskian W is identically equal to zero if the functions are linearly dependent, otherwise the functions are linearly independent.

Theorem 3.4. General solutions of homogeneous equations
Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation (Eq. (3.4))

$$y'' + p(t)y' + q(t)y = 0,$$

with p and q continuous on the open interval I. If Y is any solution whatsoever of Eq. (3.4) on I, then there exist numbers c_1 and c_2 such that

$$Y(t) = c_1 y_1(t) + c_2 y_2(t),$$

for all $t \in I$.

Remark 3: By the existence and uniqueness theorem Eq. (3.4) has two linearly independent solutions. Thus, by Theorem 3.4, if we can find two linearly independent solutions then we can find all of the solutions.

3.2 General solutions of linear equations:

Now we will generalize the previous results to nth order linear equations.

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_{n-1}(t)y' + P_n(t)y = F(t), \tag{3.8}$$

where we will assume that the functions P_i and F are continuous on an open interval I. If we further assume that $P_0(t) \neq 0$ on I then we can write Eq. (3.8) in the standard form

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = f(t), \tag{3.9}$$

where $p_i(t) = P_i(t)/P_0(t)$ and $f(t) = F(t)/P_0(t)$. Also, if $f(t) \equiv 0$ then we call

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0, \tag{3.10}$$

the associated **homogeneous** linear equation.

Theorem 3.5. Principle of Superposition
Let $y_1, \dots, y_n$ be n solutions of the homogeneous linear equation in (3.10) on the interval I. If $c_1, \dots, c_n$ are n constants, then the linear combination

$$y = c_1 y_1 + \cdots + c_n y_n,$$

is also a solution of Eq. (3.10).

Theorem 3.6. Existence and uniqueness for linear equations
Suppose that the functions p_i and f are continuous on the open interval I containing the point t_0. Then, given any n numbers $b_0, b_1, \dots, b_{n-1}$, the equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = f(t), \tag{3.11}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(t_0) = b_0, \quad y'(t_0) = b_1, \quad \dots, \quad y^{(n-1)}(t_0) = b_n. \tag{3.12}$$

Remark 4: Equation (3.11) and the conditions (3.12) constitute an nth-order linear **initial-value problem**. Theorem 3.6 tells us that any such initial-value problem has a unique solution on the *whole* interval I where the coefficient functions in (3.11) are continuous. Hence the number of initial pieces of data needed to find a particular solution is given by the order of the equation.

Definition 3.2. Linear dependence of functions
The n functions $f_1, f_2, \dots, f_n$ are said to be **linearly dependent** on the interval I provided that there exist constants c_1, c_2, ..., c_n not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0, \tag{3.13}$$

on I; that is,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

for all $x \in I$.

Thus the functions $f_1, f_2, \dots, f_n$ are linearly dependent if and only if at least one of them is a linear combination of the others. Again we can use the Wronskian to test whether a set of n functions is linearly independent or dependent., namely,

$$W[f_1, f_2, \dots, f_n](x) := \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}. \tag{3.14}$$

which is the determinant of a $n \times n$ matrix.

Theorem 3.7. Wronskian of solutions
Suppose that $y_1, y_2, \dots, y_n$ are n solutions of the homogeneous nth-order linear equation (Eq. (3.10))

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0,$$

on an open interval I where each p_i is continuous. Let

$$W := W[y_1, y_2, \dots, y_n](t).$$

(a) If $y_1, y_2, \ldots, y_n$ are linearly dependent, then $W \equiv 0$ on I.

(b) If $y_1, y_2, \ldots, y_n$ are linearly independent, then $W \neq 0$ at each point of I.

Example 3.1. Use the Wronskian to prove that the functions e^x, $\cos x$, and $\sin x$ are linearly independent on the real line.

$$
\begin{aligned}
W[e^x, \cos x, \sin x](x) &= \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} \\
&= e^x \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} e^x & -\sin x \\ e^x & -\cos x \end{vmatrix} \\
&= 2e^x
\end{aligned}
$$

Thus, since $2e^x \neq 0$ for all $x \in \mathbb{R}$ we see that the functions $\{e^x, \cos x, \sin x\}$ are linearly independent on the real line.

We can now show that, given any fixed set of n linearly independent solutions of a *homogeneous* nth-order equation, *every* (other) solution of the equation can be expressed as a linear combination of those n particular solutions.

Theorem 3.8. General solutions of homogeneous equations
Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation (Eq. (3.10))

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0,$$

on an open interval I where the p_i are continuous. If Y is any solution whatsoever of Eq. (3.10), then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t),$$

for all $t \in I$.

Remark 5: Thus, *every* solution of a homogeneous nth-order linear differential equation is a linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

of any n given linearly independent solutions. On this basis we call such a linear combination a **general solution** of the differential equation.

Next, we turn our attention to solutions of the nonhomogeneous linear equation (3.9). Suppose that y_p is any particular solution of Eq.(3.9). Suppose further that Y is another solution of Eq. (3.9). Let $y_c = Y - y_p$. Then substituting y_c into (3.9) shows that y_c solves the corresponding homogeneous linear equation (3.10). Since Y was arbitrary we arrive at the following Theorem.

Theorem 3.9. Solutions of nonhomogeneous equations
Let y_p be a particular solution of the nonhomogeneous equation (3.9) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation (3.10). If Y is any solution whatsoever of Eq. (3.9) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(t) = y_c(t) + y_p(t), \tag{3.15}$$

where $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is referred to as the complimentary solution.

Remark 6: Theorem 3.9 tells us that the general solution of the nonhomogeneous linear equation is the sum of any particular solution and the general solution of the associated homogeneous equation.

3.3 Homogeneous equations with constant coefficients:

In this section we will consider the differential equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 = 0, \tag{3.16}$$

where the a_i are constant and $a_n \neq 0$. We call (3.16) a homogeneous nth-order linear differential equation with constant coefficients. It turns out that we can solve this equation exactly.

Begin with the anastz $y(t) = e^{rt}$ where r is a constant. Plugging our ansatz into Eq. (3.16) gives

$$(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0) e^{rt} = 0,$$

which equals zero if and only if

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0. \tag{3.17}$$

We call (3.17) the characteristic equation. Thus, $y(t) = e^{rt}$ solves (3.16) if and only if r is a root of the nth degree polynomial (3.17). By the Fundamental Theorem of algebra Eq. (3.17) has exactly n solutions counting multiplicity. It turns out that we can use the roots of Eq. (3.17) to generate n linearly independent solutions of (3.16). We break the analysis up intro three cases.

Theorem 3.10. Distinct real roots
If the roots $r_1, \ldots, r_n$ of the characteristic equation (3.16) are real and distinct, then

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t} \tag{3.18}$$

is a general solution of Eq. (3.16).

Next we consider when the characteristic equation has a root of multiplicity k. Since the roots are not distinct we cannot produce n linearly independent solutions. For example, if the characteristic polynomial corresponding to a third-order equation is of the form $(r-1)^2(r+2) = 0$ then we get two independent solutions from the roots, namely, e^t and e^{-2t}. To find the general solution we must find a third linearly independent solution. The problem then is to produce the missing linearly independent solutions. To this end we will adopt 'operator notation'.

An **operator** is just a function whose input is also a function (instead of a number). For example, we can write Eq.(3.16) as the operator equation

$$Ly = 0, \tag{3.19}$$

where

$$L := a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} \cdots + a_2 \frac{d^2}{dt^2} + a_1 \frac{d}{dt} + a_0.$$

We call L a linear operator because it has the property $L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2y_2$. Now if we let $D := \frac{d}{dt}$ then we can write

$$L = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_2D^2 + a_1D + a_0, \tag{3.20}$$

which is called a **polynomial differential operator**. The point is that we can actually think of the nth-order linear equation as a formal nth degree polynomial. Further it is easy to check that $(D - a)(D - b) = (D - b)(D - a)$, i.e., the factors of the operator commute. Commutativity is the property that allows us to generate more linearly independent solutions.

Suppose the nth-order equation can be written as

$$(D - r_0)(D - r_1)^k y = Ly = 0,$$

then certainly e^{r_0t} and e^{r_1t} are solutions. Assume $y(t) = u(t)e^{r_1t}$ and try to determine $u = u(t)$ so that $u(t)e^{r_1t}$ satisfies the equation. Since $(D - r_0)(D - r_1)^k y = 0 \iff (D - r_0)y = 0$ or $(D - r_1)^k y = 0$. Next, by induction one can show

$$(D - r_1)^k[ue^{r_1t}] = (D^k u)e^{r_1t}.$$

Thus,

$$\begin{aligned}(D - r_1)^k y = 0 &\iff D^k u = 0\\ &\iff u^{(k)} = 0\\ &\iff u = c_1 + c_2t + \cdots + c_kt^{k-1}\end{aligned}$$

Hence we have the following theorem.

Theorem 3.11. Repeated roots
If the characteristic equation (3.17) has a repeated root r of multiplicity k, then the part of a general solution of the differential equation (3.16) corresponding to r is of the form

$$(c_1 + c_2t + c_3t^2 + \cdots + c_kt^{k-1})e^{rt} \tag{3.21}$$

The final case is when the characteristic polynomial has complex roots. Consider a point (x, y) in the plane. Then we can form the **complex number** $z = x + iy$, where $i = \sqrt{-1}$, and $i^2 = -1$. Further recall the Taylor series of the exponential function

$$e^t = \sum_{n=0}^{\infty} = \frac{t^n}{n!}.$$

Thus,

$$
\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} = \frac{(i\theta)^n}{n!} \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{i\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\
&= \Big(1 - \frac{\theta^2}{2!} + \frac{i\theta^4}{4!} - \cdots\Big) + i\Big(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\Big).
\end{aligned}
$$

This gives **Euler's formula**

$$
e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.22}
$$

Finally, recall the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$
\begin{aligned}
z &= x + iy \\
&= r\cos\theta + i\sin\theta \\
&= r(\cos\theta + i\sin\theta) \\
&= e^{i\theta}.
\end{aligned}
$$

Hence $z = r^{i\theta}$ is the **polar form** of a complex number. The inverse transformation is $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.

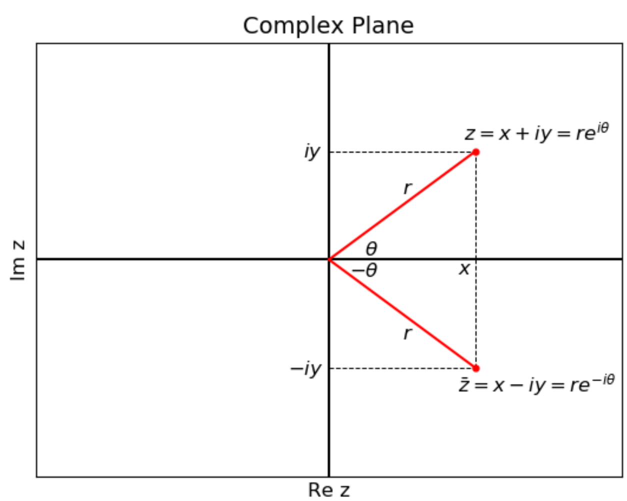

Figure 12: Complex plane and complex number $z = x + iy$. In polar form $z = re^{i\theta}$, where $r^2 = x^2 + y^2$ is the modulus and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is the argument.

Since the characteristic polynomial (3.17) is assumed to only have real coefficients all complex roots come in conjugate pairs. That is, if $z = x + iy$ is a root of (3.17), then so is the complex conjugate $\bar{z} = x - iy$. The complex conjugate is simply a reflection across the real z-axis. Also since $y(t) = e^{(a+ib)t}$ is complex-valued when $b \neq 0$ we check the derivative, i.e.,

$$\begin{aligned}\frac{d}{dt}e^{rt} = \frac{d}{dt}e^{(a+ib)t} &= \frac{d}{dt}e^{at}e^{ibt} \\ &= ae^{at}(\cos bt + i\sin bt) + e^{at}(-b\sin bt + ib\cos bt) \\ &= (a+ib)e^{at}(\cos bt + i\sin bt) \\ &= (a+ib)e^{(a+ib)t} \\ &= re^{rt},\end{aligned}$$

where we used Euler's formula and $i^2 = -1$. Thus we have the following theorem for complex roots.

Theorem 3.12. Complex roots
If the characteristic equation (3.17) has an unrepeated pair of complex conjugate roots $a \pm ib$ (with $b \neq 0$), then the corresponding part of a general solution of Eq. (3.16) has the form

$$y(t) = e^{at}(A\cos bt + B\sin bt) \tag{3.23}$$

Note that Eq. (3.23) is real-valued even though the roots were complex. If we start with the complex-valued solution $y(t) = c_1e^{rt} + c_2e^{\bar{r}t}$ then using Euler's formula and the fact that the constants are arbitrary we can express the solution in the form of (3.23).

We have now exhausted all possible cases and can thus find a general solution to a homogeneous nth-order linear equation with constant coefficients. Let's consider two examples.

Example 3.2. Solve the initial-value problem $y'' - 6y' + 25y = 0,\;\; y(0) = 3,\;\; y'(0) = 1$. This differential equation has characteristic equation given by

$$r^2 - 6r + 25 = 0.$$

Completing the square gives

$$(r-3)^2 + 16 = 0 \iff r = 3 \pm i4.$$

Thus, by Theorem 3.12 a general solution is given by

$$\begin{aligned}y(t) &= e^{3t}(A\cos 4t + B\sin 4t) \\ y'(t) &= 3e^{3t}(A\cos 4t + B\sin 4t) + e^{3t}(-4A\sin 4t + 4B\cos 4t).\end{aligned}$$

To find the particular solution consider $y(0) = A = 3$, and $y'(0) = 3A + 4B = 1 \iff B = -2$. Hence the particular solution is

$$y(t) = e^{3t}(3\cos 4t - 2\sin 4t).$$

Example 3.3. Suppose that the characteristic polynomial of a differential equation is

$$r^4(r-3)(r+5)^2((r-2)^2+9)^2.$$

Find a general solution. We have $r = 0$ is a root multiplicity 4, $r = 3$ is a root of multiplicity 1, $r = -5$ is a root of multiplicity 2, and $r = 2 \pm i3$ is a complex conjugate pair of roots of multiplicity 2. Hence, a general solution is given by

$$c_1 + c_2t + c_3t^2 + c_4t^3 + c_5e^{3t} + (c_6 + c_7t)e^{-5t} + (c_8\cos 3t + c_9\cos 3t)e^{2t} + (c_{10}\cos 3t + c_{11}\sin 3t)te^{2t}.$$

3.4 Nonhomogeneous equations and undetermined coefficients:

In this section we solve the nonhomogeneous linear equation with constant coefficients, namely,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t), \tag{3.24}$$

where a_i are constant and $a_n \neq 0$. Recall that a general solution of Eq. (3.24) is given by

$$y(t) = y_c(t) + y_p(t), \tag{3.25}$$

where y_c is a general solution of the associated homogeneous equation (3.16) and y_p is any particular solution of (3.24). Thus we need only find a single solution of (3.24) to find all solutions.

Remark 7: If the function $f = f(t)$ is an mth degree polynomial then it is reasonable to expect that there exists a solution of Eq. (3.24) that is also an mth degree polynomial. This is due to the fact that the derivative of a polynomial is a polynomial over lower degree.

Remark 8: If the function $f = f(t)$ is an exponential then it is reasonable to expect that there exists a solution of Eq. (3.24) that is also an exponential. This is due to the fact that the derivative of an exponential is some multiple of an exponential.

Remark 9: If the function $f = f(t)$ involves sine and cosines then it is reasonable to expect that there exists a solution of Eq. (3.24) that is some combination of sines and cosines. This is due to the fact that derivatives of sines and cosines are cosines and sines.

This leads us to the **method of undetermined coefficients**. The method applies whenever the function $f = f(t)$ in Eq. (3.24) is a linear combination of (finite) products of the following three types

1. A polynomial in t;
2. An exponential function e^{rt};
3. $\cos kt$ or $\sin kt$.

Any such function has the crucial property that only *finitely* many linearly independent functions appear as terms (summands) in $f(t)$ and its derivatives of all orders. In Rules 1 and 2 we assume that $Ly = f(t)$ is a nonhomogeneous linear equation with constant coefficients and that $f(t)$ is a function of this kind.

Rule 1: Method of undetermined coefficients
Suppose that no term appearing either in $f(t)$ of in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$ (no duplication). Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(t)$.

Example 3.4. Solve the initial-value problem

$$\begin{aligned} &y'' - 3y' + 2y = 3e^{-t} - 10\cos 3t; \\ &y(0) = 1, \quad y'(0) = 2. \end{aligned}$$

The characteristic equation $r^2 - 3r + 2 = 0$ has roots $r = 1$ and $r = 2$, so the complementary function is

$$y_c(t) = c_1e^t + c_2e^{2t}$$

The terms involved in $f(t) = 3e^{-t} - 10\cos 3t$ and its derivatives are e^{-t}, $\cos 3t$, and $\sin 3t$. Because none of these appears in y_c, we try

$$\begin{aligned}
y_p &= Ae^{-t} + B\cos 3t + C\sin 3t \\
y_p' &= -Ae^{-t} - 3B\sin 3t + 3C\cos 3t \\
y_p'' &= Ae^{-t} - 9B\cos 3t - 9C\sin 3t
\end{aligned}$$

After we substitute these expressions into the differential equation a collect coefficients, we get

$$\begin{aligned}
y_p'' - 3y_p' + 2y_p &= 6Ae^{-t} + (-7B - 9C)\cos 3t + (9B - 7C)\sin 3t \\
&= 3e^{-t} - 10\cos 3t
\end{aligned}$$

We equate the coefficients of the terms involving e^{-t}, those involving $\cos 3t$, and those involving $\sin 3t$. The result is the system

$$\begin{aligned}
6A &= 3, \\
-7B - 9C &= -10, \\
9B - 7C &= 0,
\end{aligned}$$

with solution $A = 1/2$, $B = 7/13$, and $C = 9/13$. This gives the particular solution

$$y_p(t) = \frac{1}{2}e^{-t} + \frac{7}{13}\cos 3t + \frac{9}{13}\sin 3t,$$

which, however, does not have the required initial values. To satisfy the initial conditions, we begin with the *general* solution

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= c_1e^t + c_2e^{2t} + \frac{1}{2}e^{-t} + \frac{7}{13}\cos 3t + \frac{9}{13}\sin 3t
\end{aligned}$$

with derivative

$$y'(t) = c_1e^t + 2c_2e^{2t} - \frac{1}{2}e^{-t} - \frac{21}{13}\sin 3t + \frac{27}{13}\cos 3t.$$

The initial conditions lead to the equations

$$\begin{aligned}
y(0) &= c_1 + c_2 + \frac{1}{2} + \frac{7}{13} = 1, \\
y'(0) &= c_1 + 2c_2 - \frac{1}{2} + \frac{27}{13} = 2,
\end{aligned}$$

with solution $c_1 = -\frac{1}{2}$, $c_2 = \frac{6}{13}$. The desired particular solution is therefore

$$y(t) = -\frac{1}{2}e^t + \frac{6}{13}e^{2t} + \frac{1}{2}e^{-t} + \frac{7}{13}\cos 3t + \frac{9}{13}\sin 3t.$$

We now turn our attention to the situation in which Rule 1 does not apply: Some of the terms involved in $f(t)$ and its derivatives satisfy the associated homogeneous equation. The strategy is to remove the 'duplication'. This can be achieved by multiplying each term of the trial solution by the least integral power of t that suffices to eliminate duplication among the trial and complementary solutions.

Before stating Rule 2 we note that to find a particular solution of the nonhomogeneous linear differential equation

$$Ly = f_1(t) + f_2(t), \tag{3.26}$$

it suffices to find *separately* particular solutions $Y_1(t)$ and $Y_2(t)$ of the two equations

$$Ly = f_1(t) \qquad Ly = f_2(t),$$

respectively. For linearity then gives

$$L(Y_1 + Y_2) = LY_1 + LY_2 = f_1(t) + f_2(t)$$

,

and therefore $y_p = Y_1 + Y_2$ is a particular solution of (3.26). Our problem is to find a particular solution of the equation $Ly = f(t)$, where $f = f(t)$ is a linear combination of products of the form

$$P_m(t)e^{rt}\cos kt, \qquad P_m(t)e^{rt}\sin kt, \tag{3.27}$$

where $P_m(t)$ is a polynomial in t of degree m.

Rule 2: Method of undetermined coefficients

If the function $f(t)$ is of either form in (3.27), take as the trial solution

$$y_p(t) = t^s\big[(A_0 + A_1t + \cdots + A_mt^m)e^{rt}\cos kt + (B_0 + B_1t + \cdots + B_mt^m)e^{rt}\sin kt\big], \tag{3.28}$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c. Then determine the coefficients in Eq. (3.28) by substituting y_p into the homogeneous equation.

Remark 10: If the nonhomogeneous equation has the form (3.26) then we take as y_p the sum of the trial solutions for f_1 and f_2, choosing s separately for each part to eliminate duplication.

Below are the substitutions used in the method of undetermined coefficients.

$f(t)$	$y_p(t)$
$P_m(t) = b_0 + b_1t + \cdots b_mt^m$	$t^s(A_0 + A_1t + \cdots + A_mt^m)$
$a\cos kt + b\sin kt$	$t^s(A\cos kt + B\sin kt)$
$e^{rt}(a\cos kt + b\sin kt)$	$t^se^{rt}(A\cos kt + B\sin kt)$
$P_m(t)e^{rt}$	$t^s(A_0 + A_1t + \cdots + A_mt^m)e^{rt}$
$P_m(t)(a\cos kt + b\sin kt)$	$t^s\big[(A_0 + A_1t + \cdots + A_mt^m)\cos kt$ $+(B_0 + B_1t \cdots + B_mt^m)\sin kt\big]$

Example 3.5. Determine the appropriate form for a particular solution of the fifth-order equation

$$(D-2)^3(D^2+9)y = t^2e^{2t} + t\sin 3t.$$

First note that the complementary solution is

$$y_c(t) = (c_1 + c_2 t + c_3 t^2)e^{2t} + c_4 \cos 3t + c_5 \sin 3t.$$

Next, consider $t^2 e^{2t}$ for which we write the general form $(A + Bt + Ct^2)e^{2t}$. Also, the term $t \sin 3t$ has the generalized form $(D + Et) \cos 3t + (F + Gt) \sin 3t$. Thus, to remove the duplication we multiply the first expression by t^3 and the second expression by t. Thus,

$$y_p(t) = t^3(A + Bt + Ct^2)e^{2t} + t(D + Et) \cos 3t + t(F + Gt) \sin 3t.$$

Finally, let's consider a case where the method of undetermined coefficients cannot be used. The differential equation

$$y'' + y = \tan t$$

cannot be solved using the method of undetermined coefficients. This is because the function $f(t) = \tan t$ has *infinitely* many linearly independent derivatives. Therefore, we do not have a *finite* linear combination available to use as a trial solution.

The method of **variation of parameters** can in principle (if the integrals can be evaluated) always be used to find a particular solution of the nonhomogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = f(t), \tag{3.29}$$

provided we know the complementary solution, i.e., a general solution of the associated homogeneous equation

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t). \tag{3.30}$$

Here, in brief, is the basic idea of the method of variation of parameters. Suppose that we replace the constants, or *parameters*, $c_1, \ldots, c_n$ in the complementary function (3.30) with *variables*: functions $u_1, u_2, \ldots, u_n$ of t. The problem is to choose these functions in such a way that the combination

$$y_p(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$$

is a particular solution of the nonhomogeneous equation (3.29). It turns out this is always possible. Here we provide the solution for the second-order linear equation.

Theorem 3.13. Variation of parameters
If the nonhomogeneous equation $y'' + p(t)y' + q(t)y = f(t)$ has complementary function $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$, then a particular solution is given by

$$y_p(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(t)} dt + y_2(t) \int \frac{y_1(t)f(t)}{W(t)} dt. \tag{3.31}$$

where $W(t) = W[y_1, y_2](t)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

3.5 Mechanical Vibrations:

We end this section with an application of linear differential equations with constant coefficients. The motion of a mass attached to a spring serves as a relatively simple example of the vibrations that occur in more complex mechanical systems.

Consider a body of mass m attached to one end of an ordinary spring that resists compression as well as stretching; the other end of the spring is attached to a fixed wall.

Denote by x the displacement from equilibrium (resting) position. We take $x > 0$ when the spring is stretched, and thus $x < 0$ when it is compressed. To derive a differential equation that describes the motion of the mass as a function of time we apply Newton's second law of motion, i.e., $F = ma$. Let's begin by considering the various forces that act on the body.

According to Hooke's law, the restorative force F_S that the spring exerts on the mass is proportional to the distance x that the spring is stretched. Thus,

$$F_S = -kx,$$

where $k > 0$ is referred to as the **spring constant**. Next, we assume that there is a frictional force that is proportional to velocity. Thus,

$$F_R = -cx',$$

where $c > 0$ is referred to as the **damping constant**. Finally denote and force external to the system by F_E. Thus, we have

$$mx'' = F_S + F_R + F_E,$$

which can be written as

$$mx'' + cx' + kx = F_E(t), \tag{3.32}$$

a nonhomogeneous linear differential equation with constant coefficients. Note that when $F_E(t) \equiv 0$, i.e., no external force we say the motion is 'free'. Also, if $c = 0$, i.e., no friction we say the motion is 'undamped'.

As an example of a system governed by (3.32) consider the **simple pendulum**. Suppose you have a mass m swinging back and forth on the end of a string (or *massless rod*) of length L as shown in Fig. 13. We may specify the position of the mass at time t by giving the counterclockwise angle $\theta = \theta(t)$ that the string or rod makes with the vertical at time t. To analyze the motion of the mass m, we will apply the law of *conservation of mechanical energy*, according to which the sum of the kinetic energy and the potential energy remains constant. Recall that the arclength between two points on a circle of radius r is given by $\frac{\theta}{2\pi} \cdot 2\pi r$. Thus, the position of the mass is given by $s = L\theta$. This implies that the velocity of the mass is $s' = L\theta'$, therefore its kinetic energy is

$$\frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

Next, choosing the lowest point attained by the mass O as a reference point we calculate the potential energy. Recall that a vector field $\mathbf{F}$ is conservative if it is the gradient of a scalar potential function, i.e., $\mathbf{F} = -\nabla V$. Thus, the potential energy is given by

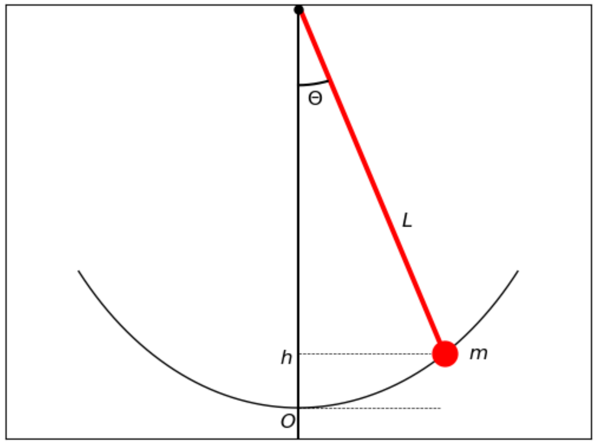

Figure 13: The simple pendulum.

$$mg\int_0^h dy = mgh = mgL(1-\cos\theta).$$

Notice that $\cos\theta = \frac{L-h}{L}$ which gives the potential energy in terms of θ. Thus, the total energy in the system is given by

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta) = E.$$

Taking derivatives of both sides of the above equation we have $mL^2\theta'\theta'' + mgL\sin\theta\theta' = 0 \iff mL\theta'(L\theta'' + g\sin\theta) = 0$. Thus, we have the equation of motion for the simple pendulum, namely,

$$\theta'' + \frac{g}{L}\sin\theta = 0. \tag{3.33}$$

Finally, assuming θ is small we have $\sin\theta \approx \theta$ and so

$$\theta'' + k\theta = 0, \qquad k = g/L, \tag{3.34}$$

which is the same equation derived for the spring-mass system without the damping constant.
Next, we break our analysis of mechanical vibrations into different cases.

i Free/Undamped motion ($F_E \equiv 0, c = 0$).

ii Free/Damped motion ($F_E \equiv 0, c \neq 0$).

iii Undamped/Forced motion ($F_E \neq 0, c = 0$).

Free/Undamped motion: In this case equation (3.32) is given by

$$x'' + \omega_0^2 x = 0, \qquad \omega_0 = \sqrt{k/m}$$

which can be solved easily since it is a homogeneous linear equation with constant coefficients. Thus, the characteristic equation is given by $r^2 + \omega_0^2 = 0$ which has roots $r = \pm i\omega_0$. Hence a general solution is given by

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t.$$

Using right triangle trigonometry we can rewrite the general solution as

$$x(t) = C\cos(\omega_0 t - \alpha),$$

where $C = \sqrt{A^2 + B^2}$, $\cos\alpha = A/C$, $\sin\alpha = B/C$, and $\alpha = \arctan(B/A)$.

Thus, the mass oscillates about its equilibrium positions with

- Amplitude C
- Circular Frequency ω_0 (radians/second)
- Phase angle α
- Period $T = \frac{2\pi}{\omega_0}$ (seconds)
- Frequency $\nu = \frac{1}{T}$ (hertz)
- Time lag $\delta = \frac{\alpha}{\omega_0}$

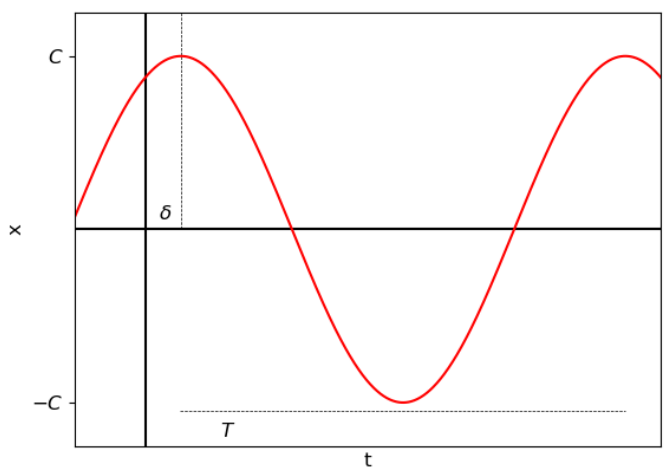

Figure 14: Harmonic oscillator $x(t) = C\cos(\omega_0 t - \alpha)$.

Free/Damped motion: In this case equation (3.32) is given by

$$x'' + 2px' + \omega_0^2 x = 0, \qquad \omega_0 = \sqrt{k/m}, \qquad p = \frac{c}{2m},$$

which can be solved easily since it is a homogeneous linear equation with constant coefficients. Here the characteristic equation is given by $r^2 + 2pr + \omega_0^2$ which has roots $r_1, r_2 = -p \pm (p^2 - \omega_0^2)^{1/2}$ that depend on the sign of

$$p^2 - \omega_0^2 = \frac{c^2 - 4km}{4m^2}.$$

Thus, the analysis can be broken up into three different cases classified by the root of the characteristic polynomial. First, if $c^2 > 4km$ (**overdamped**) then there are two distinct real roots that are negative. Thus

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

and $x(t) \to 0$ as $x \to \infty$. In this case there is strong frictional force that 'damps' out any oscillation. Next, if $c^2 = 4km$ (**critically damped**) then there is one real root $r = -p$. Thus

$$x(t) = (c_1 + c_2 t)e^{-pt},$$

and $x(t) \to 0$ as $t \to \infty$. Here there is just enough frictional force to damp out any oscillations; and the mass crosses the equilibrium position at most one time. Finally, if $c^2 < 4km$ (**underdamped**) then there are complex conjugate roots. Thus,

$$x(t) = e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t),$$

where $\omega_1 = \sqrt{4km - c^2}/2m$. Like before we can rewrite this equation as

$$Ce^{-pt} \cos(\omega_1 t - \alpha).$$

In this case a small frictional force allows the mass to oscillate about its equilibrium position. Indeed one can think of Ce^{-pt} as a time-varying amplitude; and $T_1 = 2\pi/\omega_1$ can be thought of as a 'pseudoperiod'. In this case we also have $x(t) \to 0$ as $t \to \infty$. See Fiq. 15 for a graphical presentation of each case.

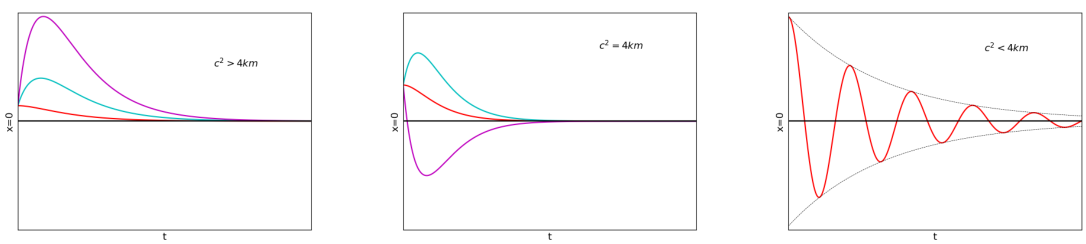

Figure 15: (Left) Overdamped, $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. (Center) Critically damped, $x(t) = (c_1 + c_2 t)e^{-pt}$. (Right) Underdamped, $x(t) = Ce^{-pt} \cos(\omega_1 t - \alpha)$. Also, $A(t) = \pm Ce^{-pt}$ is the time-varying amplitude, or the envelope curve.

Undamped/Forced motion: In this case equation (3.32) is given by

$$mx'' + kx = F_0 \cos \omega t,$$

where $F_0 \cos \omega t = F_E(t)$ is the external forcing function. Note that this external force is assumed to be periodic. Thus, we can use the method of undetermined coefficients to solve this equation. First, note that the associated homogeneous equation has complementary solution given by $x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $\omega_0 = \sqrt{k/m}$. So long as $\omega \neq \omega_0$ (no duplication) we take as a trial solution $x_p(t) = A \cos \omega t$. This gives

$$-m\omega^2 A\cos\omega t + kA\cos\omega t = F_0\cos\omega t,$$

so

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2},$$

and thus

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t.$$

If we impose the initial conditions $x(0) = x'(0) = 0$ on the solution $x = x_c + x_p$ we find that the particular solution is given by

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t).$$

Using the trigonometric identity $2\sin A\sin B = \cos(A-B) - \cos(A+B)$, with $A = \frac{1}{2}(\omega_0 + \omega)t$, and $B = \frac{1}{2}(\omega_0 - \omega)t$, enable us to write the particular solution $x = x(t)$ in the form

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\frac{1}{2}(\omega_0 - \omega)t\sin\frac{1}{2}(\omega_0 + \omega)t.$$

If $\omega_0 \approx \omega$ then $|\omega_0 - \omega|$ is small relative to $|\omega_0 + \omega|$. Thus $\sin\frac{1}{2}(\omega_0 - \omega)t$ is slowly-varying while $\sin\frac{1}{2}(\omega_0 + \omega)t$ is rapidly-varying. We can consider $A(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\frac{1}{2}(\omega_0 - \omega)t$ a slowly-varying periodic amplitude function that serves as an envelope curve for the rapidly-varying $\sin\frac{1}{2}(\omega_0 + \omega)t$. A rapid oscillation with a (comparatively) slowly-varying periodic amplitude exhibits the phenomenon of **beats**.

Looking at $x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t$ we see that the amplitude $A = \frac{F_0/m}{\omega_0^2 - \omega^2}$ is large when the natural and external frequencies ω_0 and ω are approximately equal. Let's rewrite the amplitude as follows:

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/k}{1 - (\omega/\omega_0)^2} = \pm\frac{\rho F_0}{k},$$

where F_0/k is the **static displacement** of a spring with constant k due to a *constant* force F_0, and the **amplification factor** ρ is defined to be

$$\rho = \frac{1}{|1 - (\omega/\omega_0)^2|}.$$

It is clear that $\rho \to \infty$ as $\omega \to \omega_0$. This is the phenomenon of **resonance**— the increase without bound (as $\omega \to \omega_0$) in the amplitude of oscillations of an undamped system with natural frequency ω_0 in response to an external force with frequency $\omega \approx \omega_0$. We have been assuming that $\omega \neq \omega_0$ (no duplication). What happens if we assume $\omega = \omega_0$ (duplication)? In this case we take

$$x_p(t) = t(A\cos\omega_0 t + B\sin\omega_0 t),$$

as the trial solution and find that $A = 0$, and $B = F_0/(2m\omega_0)$. Hence the particular solution is

$$x_p(t) = \frac{F_0}{2m\omega_0}t\sin\omega_0 t.$$

Thus, in the case of *pure resonance* , i.e., $\omega = \omega_0$ we have that the amplitude of the oscillations grow without bound.

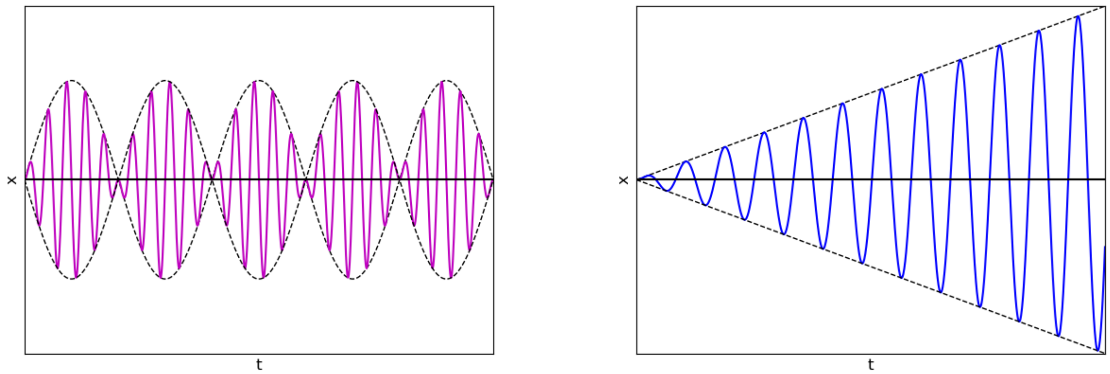

Figure 16: (Left) Rapidly varying oscillation with slowly varying envelope curve, i.e., beats. (Right) Amplitude of the oscillations grow without bound, i.e., resonance.

Example 3.6. With $m = 0.1$, $F_0 = 50$, $\omega_0 = 55$, and $\omega = 45$ there is no duplication. We are in the undamped/forced oscillations case and thus the solution is given by

$$x(t) = \sin 5t \sin 50t$$

This solution has a rapid oscillation of frequency $\frac{1}{2}(\omega_0 + \omega) = 50$ that is 'modulated' by the amplitude function $A(t) = \sin 50t$ of frequency $\frac{1}{2}(\omega_0 - \omega) = 5$.

Damped/Forced motion: In real physical systems there is always some damping, from frictional effects if nothing else. The complementary function x_c of the equation

$$mx'' + cx' + kx = F_0 \cos \omega t \tag{3.35}$$

is given by

$$\begin{aligned} x_c(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} && \text{(overdamped, } c^2 > 4km) \\ x_c(t) &= (c_1 + c_2 t)e^{-pt} && \text{(critically dampled, } c^2 = 4km) \\ x_c(t) &= Ce^{-pt}\cos(\omega_1 t - \alpha) && \text{(underdamped, } c^2 < 4km) \end{aligned}$$

and the key point is that in all cases $x_c(t) \to 0$ as $t \to \infty$. Thus x_c is a **transient solution** of Eq. (3.35)—one that become negligible with the passage of time, leaving only the particular solution x_p.

We take $x_p(t) = A\cos \omega t + B \sin \omega t$ as a trial solution and apply the method of undetermined coefficients. This gives

$$x_p(t) = C\cos(\omega t - \alpha), \qquad \text{(steady periodic motion)}$$

where $C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 - (c\omega)^2}}$, and $\tan \alpha = \frac{c\omega}{k - m\omega^2}$ with $0 < \alpha < \pi$. Note that if $c > 0$, then the *forced amplitude* always remains finite, in contrast with the case of resonance in the undamped case when the forcing frequency ω equals the natural frequency $\omega_0 = \sqrt{k/m}$. But the forced amplitude may attain a maximum for some value of ω, in which case we speak of *practical resonance*. To see if and when practical resonance occurs, we need only graph C as a function of ω and look for a global maximum. Finally, we note that the solution can be decomposed into two pieces, i.e.,

$$x(t) = x_{\text{tr}}(t) + x_{\text{sp}}(t) \tag{3.36}$$

where x_{tr} is the transient solution which dies out in time; and x_{sp} is the steady periodic solution due to external forcing.

4 Linear systems of differential equations:

In this section we will study systems of differential equations. Specifically we will be interested in explicit first-order systems with n dependent variables and one independent variable.

$$\begin{aligned}
x_1' &= f_1(t, x_1, \ldots, x_n) \\
x_2' &= f_2(t, x_1, \ldots, x_n) \\
&\vdots \\
x_n' &= f_n(t, x_1, \ldots, x_n)
\end{aligned}$$

The first thing to note is that any explicit nth-order differential equation

$$x^{(n)} = f(t, x, x', x'', \ldots, x^{(n-1)})$$

can be written as an equivalent system of n first-order differential equations using the following algorithm. First, make the substitutions

$$\begin{aligned}
x_1 &= x \\
x_2 &= x' \\
x_3 &= x'' \\
&\vdots \\
x_n &= x^{(n-1)}
\end{aligned}$$

then taking derivatives gives $x_1' = x' = x_2$, $x_2' = x'' = x_3$, and so forth. Doing this for each of the n dependent variables gives the following system of n differential equations.

$$\begin{aligned}
x_1' &= x_2 \\
x_2' &= x_3 \\
&\vdots \\
x_{n-1}' &= x_n \\
x_n' &= f(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

Example 4.1. Rewrite the second-order linear differential equation

$$y'' + p(t)y' + q(t)y = f(t)$$

as an equivalent system of differential equations with two dependent variables.

We begin by noting that we can rewrite this differential equation in explicit form, i.e., $y'' = -p(t)y' - q(t)y + f(t)$. Next make the substitutions $x_1 = y$ and $x_2 = y'$. Then taking derivatives gives $x_1' = y' = x_2$ and $x_2' = y'' = -p(t)y' - q(t)y + f(t) = -p(t)x_2 - q(t)x_1 + f(t)$. Thus, we have a system of two differential equations.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -p(t)x_2 - q(t)x_1 + f(t) \end{aligned}$$

In general a linear system of first-order differential equations is given by

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + f_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + f_n(t) \end{aligned} \tag{4.1}$$

Soon we will see how to write (4.1) in a compact form using matrices. We say that the linear system (4.1) is **homogeneous** if $f_i(t) \equiv 0$ for $i = 1, \ldots, n$. Otherwise we say the system is nonhomogeneous. A **solution** of the system (4.1) is an n-tuple of functions $x_1(t), x_2(t), \ldots, x_n(t)$ that (on some interval) identically satisfy each of the equations in (4.1). We will see that the general theory of a system of n linear first-order equations shares many similarities with the general theory of a single nth-order linear differential equation.

Theorem 4.1. Suppose that the functions $p_{11}, p_{12}, \ldots, p_{nn}$ and the functions $f_1, f_2, \ldots, f_n$ are continuous on the open interval I containing the point a. Then, given the n numbers $b_1, b_2, \ldots, b_n$, the system (4.1) has a unique solution on the entire interval I that satisfies the n initial conditions

$$x_1(a) = b_1, \quad x_2(a) = b_2, \quad \ldots, \quad x_n(a) = b_n$$

Graphically a solution of (4.1) is given by a parametric curve, and is commonly referred to as a **trajectory**. For each value of $t \in I$ the solution is an n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and referred to as the 'state' of the dynamical system at time t. Further, the set of all possible states is referred to as the **phase space**. Also, $(x_1', x_2', \ldots, x_n')$ is the **tangent vector** to the trajectory and thus points in the direction of motion. Plotting the tangent vector at each point in phase space is called a **direction field**. Finally, a picture showing the system's trajectories in phase space is called a **phase portrait**.

Example 4.2. Solve the system

$$\begin{aligned} x' &= -2y \\ y' &= \frac{1}{2}x \end{aligned}$$

Taking the derivative of the first equation gives $x'' = -2y' \iff x'' = -2(\frac{1}{2}x) \iff x'' + x = 0$. This is a homogeneous linear second-order equation with constant coefficients. Thus, since the characteristic equation is $r^2 + 1 = 0$ the general solution is given by $x(t) = A\cos t + B\sin t = C\cos(t - \alpha)$. Next, since we know x it follows that $y = -\frac{1}{2}x' = \frac{C}{2}\sin(t - \alpha)$. Thus, the solution is given by the tuple

$$(x(t), y(t)) = (C\cos(t - \alpha), C\sin(t - \alpha)/2).$$

To understand the solution's motion in the phase plane we use the identity $\sin^2\theta + \cos^2\theta = 1$. This gives

$$\frac{x^2}{C^2} + \frac{y^2}{(C/2)^2} = 1$$

which is the equation of an ellipse. Thus, the motion is counterclockwise along ellipses in the phase plane. See Fig. 17 for a plot of the direction field and phase portrait for this dynamical system.

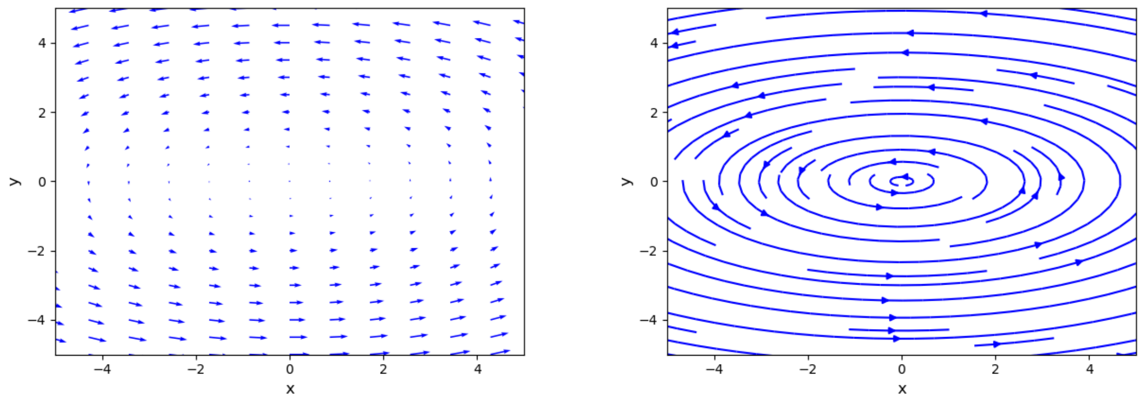

Figure 17: (Left) Direction field for the system in Example 4.2. (Right) Phase portrait for the system in Example 4.2.

The method used to solve the system of differential equations in Example 4.2 is known as **elimination**. It is an easy method for solving systems with few equations.

4.1 Elimination:

Here we discuss the method of elimination for solving a system of differential equations.

Example 4.3. Solve the system of differential equations given by

$$x' = 4x - 3y$$
$$y' = 6x - 7y$$

(Step 1) Solve the second equation for x and take its derivative.

$$x = \frac{1}{6}y' + \frac{7}{6}y \iff x' = \frac{1}{6}y'' + \frac{7}{6}y'$$

(Step 2) Plug x and x' into the first equation.

$$\frac{1}{6}y'' + \frac{7}{6}y' = 4(\frac{1}{6}y' + \frac{7}{6}y) - 3y \iff y'' + 3y' - 10y = 0$$

(Step 3) Solve the homogeneous second-order linear equation for y.

$$r^2 + 3r - 10 = 0 \iff (r+5)(r-2) = 0$$

Thus, the general solution is given by $y(t) = c_1e^{-5t} + c_2e^{2t}$.

(Step 4) Use the solution $y = y(t)$ to and the equation $x = \frac{1}{6}y' + \frac{7}{6}y$ to find $x = x(t)$.

$$x = \frac{1}{6}(-5c_1e^{-5t} + 2c_2e^{2t}) + \frac{7}{6}(c_1e^{-5t} + c_2e^{2t}) \iff x = \frac{1}{3}c_1e^{-5t} + \frac{3}{2}c_2e^{2t}$$

Thus, the solution is

$$(x(t), y(t)) = \Big(\frac{1}{3}c_1e^{-5t} + \frac{3}{2}c_2e^{2t}, c_1e^{-5t} + c_2e^{2t}\Big)$$

$$= c_1\Big(\frac{1}{3}, 1\Big)e^{-5t} + c_2\Big(\frac{3}{2}, 1\Big)e^{2t}$$

which is a linear combination of two independent solutions.

A systematic elimination procedure can be attained by using the operator notation discussed earlier. Recall the **polynomial differential operator**

$$L := a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0. \tag{4.2}$$

If L_1 and L_2 are two such operators, then their product L_1L_2 is defined as

$$L_1L_2[x] = L_1[L_2x].$$

Thus,

$$\begin{aligned}(D + a)(D + b)[x] = (D + a)[Dx + bx] &= D[Dx + bx] + a[Dx + bx]\\ &= D^2x + bDx + aDx + abx\\ &= D^2x + (a + b)Dx + abx\\ &= (D^2 + (a + b) + ab)[x]\end{aligned}$$

so when the coefficients are constant the polynomial differential operators are commutative and behave like ordinary polynomials. Thus,

$$L_1L_2[x] = L_2L_1[x]$$

if the necessary derivative of $x = x(t)$ exist. Note that in the case of variable coefficients commutativity does not hold in general.

Any system of two linear differential equations with constant coefficients can be written in the form

$$\begin{aligned}L_1x + L_2y &= f_1(t),\\ L_3x + L_4y &= f_2(t).\end{aligned} \tag{4.3}$$

where L_1, L_2, L_3, and L_4 are polynomial differential operators (perhaps different orders). For instance the system from Example 4.3 can be written in the form

$$\begin{aligned}(D - 4)x + 3y &= 0,\\ -6x + (D + 7)y &= 0.\end{aligned}$$

Let $L_1 = (D - 4), L_2 = 3, L_3 = -6$, and $L_4 = (D + 7)$. We can eliminate the dependent variable x from the system if we multiply the first equation by L_3 and the second equation by L_1. This gives,

$$L_3L_1x + L_3L_2y = L_3f_1(t),$$
$$L_1L_3x + L_1L_4y = L_1f_2(t).$$

Subtraction of the first from the second of these equations yields the single equation

$$(L_1L_4 - L_2L_3)y = L_1f_2(t) - L_3f_2(t)$$

in the single dependent variable y. Note that $L_3L_1 = L_1L_3$ since polynomial differential operators with constant coefficient are commutative. After solving for $y = y(t)$ we can substitute the result into either of the original equations and then solve for $x = x(t)$. The operator $L_1L_4 - L_2L_2$ on the left hand side of the above equation is called the **operational determinant**

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = L_1L_4 - L_2L_3. \tag{4.4}$$

Theorem 4.2. If the operational determinant (4.4) is *not identically zero*, then the number of independent arbitrary constants in a general solution of the system in (4.3) is equal to the order of its operational determinant— that is, its degree as a polynomial in D.

If the operational determinant is identically zero, then the system is said to be **degenerate**. A degenerate system may have either no solution or infinitely many independent solutions. Roughly speaking, every degenerate system is equivalent to either an inconsistent (no solution) system or a redundant (infinitely many solutions) system.

4.2 Eigenvalue method:

Consider the following object:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = [a_{ij}], \tag{4.5}$$

where m is the number of rows, n is the number of columns, and a_{ij} (element in the ith row and jth column) is a real or complex number. In this case we call A an $m \times n$ matrix. More generally we can assume $t \mapsto A(t)$ a 'matrix-valued' function, i.e.,

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}(t) & p_{m2}(t) & \dots & p_{mn}(t) \end{pmatrix} = [p_{ij}(t)], \tag{4.6}$$

where $t \in \mathbb{R}$ and each $p_{ij}(t)$ is a real-valued or complex-valued function. Suppose A and B are $m \times n$ matrices. Then:

$$\text{Addition} \qquad A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}],$$
$$\text{Scalar multiplication} \qquad cA = c[a_{ij}] = [ca_{ij}],$$

thus these operations are done elementwise. Next, suppose A is $m \times p$ matrix and B is $p \times n$ matrix. Then

$$\text{Multiplication} \qquad AB = \sum_{k=1}^{n} a_{ik} b_{kj} = [c_{ij}] = C,$$

where C is $m \times n$ matrix.

Remark 1: Multiplication of matrices, i.e., AB is defined when the number of columns of A equals the number of row of B.

Remark 2: Matrix multiplication is **not** commutative. So, in general $AB \neq BA$.

We denote $\mathbb{I}$ the identity matrix where $a_{ij} = 1$ when $i = j$ and 0 otherwise. For example,

$$\mathbb{I}_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Further we define the matrix transpose. Suppose $A = [a_{ij}]$ is $m \times n$ matrix. Then $A^T = [a_{ji}]$ is $n \times m$ matrix. For example,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Remark 3: Notice that the diagonal elements of a matrix, i.e., a_{ii} are invariant under the transpose operation.

Finally, a **row vector** is a $1 \times n$ matrix and a **column vector** is a $n \times 1$ matrix.

$$\mathbf{v} = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

Usually vectors are denoted by boldface lower case letters and matrices by uppercase letters.

Let's see how matrices can be used to write linear systems compactly. Recall that a linear system of differential equations is given by the coupled set of expressions:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + f_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + f_n(t) \end{aligned}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Thus, we can write a linear system of differential equations compactly as

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t). \tag{4.7}$$

Further, a homogeneous linear system with constant coefficients is given by

$$\mathbf{x}' = A\mathbf{x}, \tag{4.8}$$

and the corresponding initial-value problem is

$$\mathbf{x}' = A\mathbf{x}, \ \ \mathbf{x}(0) = \mathbf{x}_0. \tag{4.9}$$

Remark 4: We will only consider $n \times n$ systems, i.e., $P(t)$, and A, will always be square matrices.

Definition 4.1. Let A be $n \times n$ matrix. A nonzero vector $\mathbf{v}$ with the property

$$A\mathbf{v} = \lambda\mathbf{v}$$

is called an **eigenvector** and the scalar λ is the corresponding **eigenvalue**. Thus the matrix A maps the vector $\mathbf{v}$ to a scalar multiple of itself.

How does one find eigenvalues and eigenvectors?

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\iff A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \\ &\iff (A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0} \end{aligned}$$

Notice that $A - \lambda\mathbb{I}$ is a $n \times n$ matrix. Thus we need to find values of λ such that the linear system of equations $(A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$. This leads to the following theorem from linear algebra.

Theorem 4.3. Suppose A is $n \times n$ matrix. Then the linear system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if det$(A) = 0$.

Thus to find eigenvalues we must find λ's such that det$(A - \lambda\mathbb{I}) = 0$. Then to find the corresponding eigenvector(s) we solve the system $(A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$ where λ is now a known quantity.

Remark 5: Suppose A is $n \times n$ matrix. Then det$(A - \lambda\mathbb{I}) = 0 \iff p(\lambda) = 0$ where $p(\lambda)$ is an nth degree polynomial called the **characteristic polynomial**.

How do eigenvalues correspond to systems of differential equations?

Consider the homogeneous linear system of differential equations with constant coefficients, i.e., $\mathbf{x}' = A\mathbf{x}$. Suppose $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$. Then

$$\begin{aligned} \mathbf{x}' &= \lambda \mathbf{v} e^{\lambda t} \\ &= A\mathbf{x} \\ &= A\mathbf{v} e^{\lambda t} \end{aligned}$$

Thus,

$$\mathbf{x}' = A\mathbf{x} \iff A\mathbf{v} = \lambda \mathbf{v},$$

and so $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ is a solution to the homogeneous system if and only if λ is an eigenvalue and $\mathbf{v}$ a corresponding eigenvector of the $n \times n$ coefficient matrix A.

Like in the case of nth order linear differential equations in order to find a general solution we need to find enough linearly independent solutions.

Remark 6: Eigenvectors corresponding to different eigenvalues are linearly independent.

Remark 7: To each eigenvalue corresponds *at least* one eigenvector. An eigenvalue of multiplicity $k \geq 1$ that has k independent eigenvectors is called **complete**. Otherwise the eigenvalue is called **defective**. If the eigenvalue has multiplicity k and generates p independent eigenvectors then we define the **defect** as $d = k - p$.

Case (i) The $n \times n$ matrix A has n distinct real eigenvalues. Then there exists n linearly independent eigenvectors and a general solution of (4.8) is given by

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t}.$$

Case (ii) There exists a complex conjugate pair of eigenvalues $\lambda = a \pm ib$ of multiplicity one. Notice that

$$\begin{aligned} A\mathbf{v} = \lambda \mathbf{v} &\iff \overline{A\mathbf{v}} = \overline{\lambda \mathbf{v}} \\ &\iff A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}} \end{aligned}$$

so that if λ is an eigenvalue with eigenvector $\mathbf{v}$ then $\bar{\lambda}$ is also an eigenvalue with eigenvector $\bar{\mathbf{v}}$. The part of the solution corresponding to the complex conjugate pair of eigenvalues is then given by

$$\mathbf{x}(t) = c_1 \mathbf{v} e^{\lambda t} + c_2 \bar{\mathbf{v}} e^{\bar{\lambda} t}.$$

Notice that this solution is complex-valued since $\lambda = a + ib$. This is not ideal. So let's use superposition to try an rewrite the solution in terms of real-valued functions. Consider one of the independent solutions, i.e,

$$\begin{aligned} \mathbf{x}(t) = \mathbf{v} e^{\lambda t} &= (\mathbf{p} + i\mathbf{q}) e^{(a+ib)t} \\ &= (\mathbf{p} + i\mathbf{q}) e^{at} (\cos bt + i \sin bt) \\ &= \mathbf{p} \cos bt e^{at} + i\mathbf{p} \sin bt e^{at} + i\mathbf{q} \cos bt e^{at} - \mathbf{q} \sin bt e^{at} \\ &= e^{at} (\mathbf{p} \cos bt - \mathbf{q} \sin bt) + i e^{at} (\mathbf{p} \sin bt + \mathbf{q} \cos bt) \end{aligned}$$

Thus, taking the real and imaginary parts of the solution gives

$$\mathbf{x}(t) = c_1 e^{at}(\mathbf{p}\cos bt - \mathbf{q}\sin bt) + c_2 e^{at}(\mathbf{p}\sin bt + \mathbf{q}\cos bt).$$

Case (iii) There exists an eigenvalue of multiplicity $k > 1$ that generates $p = k$ eigenvectors. Then $d = k - p = 0$. In this case the eigenvalue is complete and thus we can generate k linearly independent solutions. Thus,

$$\mathbf{x}(t) = (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)e^{\lambda t}.$$

Case (iv) There exists an eigenvalue of multiplicity $k > 1$ that generates $p < k$ eigenvectors. Then the defect is $d = k - p \geq 1$. In this case the eigenvalue is defective and thus we must find d **generalized eigenvectors** to generate enough independent solutions.

A fundamental theorem of linear algebra states that every $n \times n$ matrix A has n linearly independent generalized eigenvectors. These n generalized eigenvectors may be arranged in chains, with the sum of the lengths of the chains associated with a given eigenvalues λ equal to the multiplicity of λ. But the structure of these chains depends on the defect of λ, and can be quite complicated. For example, a multiplicity 4 eigenvalue can correspond to:

- Four length 1 chains (defect 0);
- Two length 1 chains and a length 2 chain (defect 1);
- Two length 2 chains (defect 2);
- A length 1 chain and a length 3 chain (defect 2); or
- A length 4 chain (defect 3).

ALGORITHM: Chains of Generalized Eigenvectors

Begin with a nonzero solution $\mathbf{u}_1$ of $(A - \lambda\mathbb{I})^{d+1}\mathbf{u} = \mathbf{0}$ and successively multiply by the matrix $A - \lambda\mathbb{I}$ until the zero vector is obtained. If

$$(A - \lambda\mathbb{I})\mathbf{u}_1 = \mathbf{u}_2 \neq \mathbf{0},$$
$$\vdots$$
$$(A - \lambda\mathbb{I})\mathbf{u}_{k-1} = \mathbf{u}_k \neq \mathbf{0},$$

but $(A - \lambda\mathbb{I})\mathbf{u}_k = \mathbf{0}$, then the vectors

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{\mathbf{u}_k, \ldots, \mathbf{u}_1\}$$

(listed in reverse order of their appearance) form a length k chain of generalized eigenvectors based on the (ordinary) eigenvector $\mathbf{v}_1$.

Then a set of k independent solutions is given by:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t},$$
$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t}$$
$$\vdots$$
$$\mathbf{x}_k(t) = \left(\frac{\mathbf{v}_1 t^{k-1}}{(k-1)!} + \cdots + \frac{\mathbf{v}_{k-2} t^2}{2!} + \mathbf{v}_{k-1} t + \mathbf{v}_k \right) e^{\lambda t}$$

Finally we will show how the eigenvalues and eigenvectors of the matrix A can be used to sketch the phase portrait.

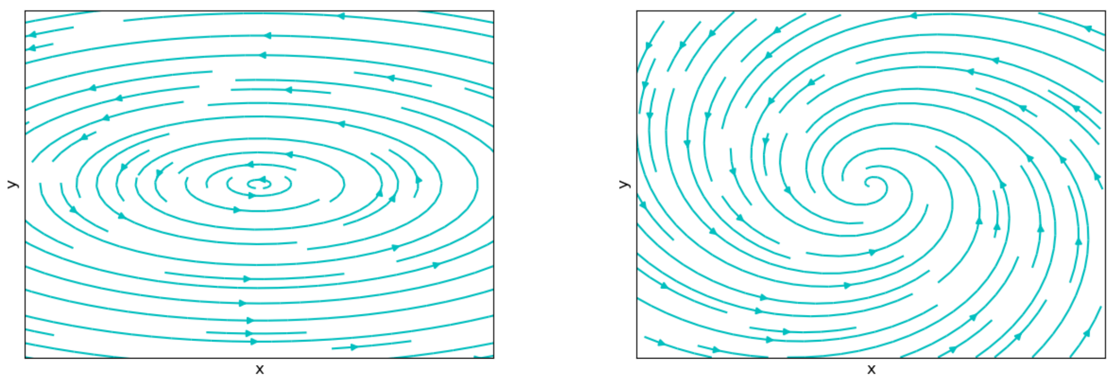

Figure 18: (Left) Center - $\lambda_{1,2} = \pm ib$. (Right) Spiral sink - $\lambda_{1,2} = a \pm ib$.

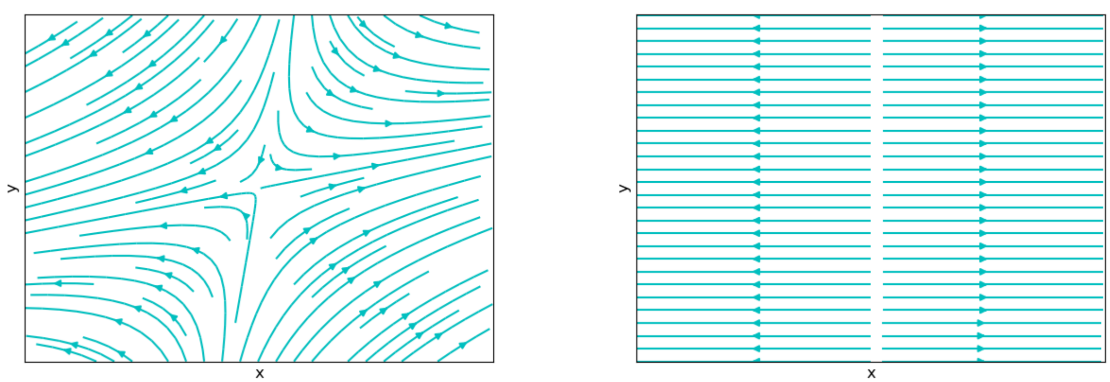

Figure 19: (Left) Saddle - $\lambda_1 < 0 < \lambda_2$. (Right) Zero Eigenvalue - $\lambda_1 = 0 < \lambda_2$.

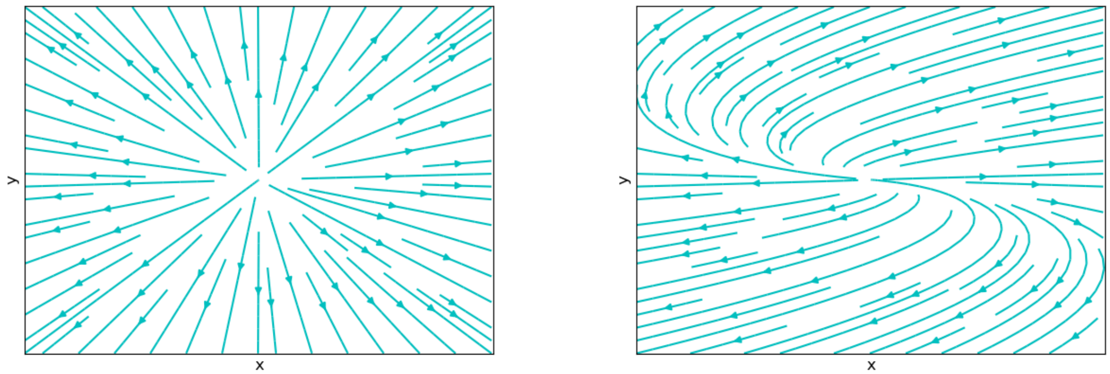

Figure 20: (Left) Proper Nodal Source - $\lambda_1 = \lambda_2$. (Right) Improper Nodal Source (one eigenvector) - $\lambda_1 = \lambda_2$.

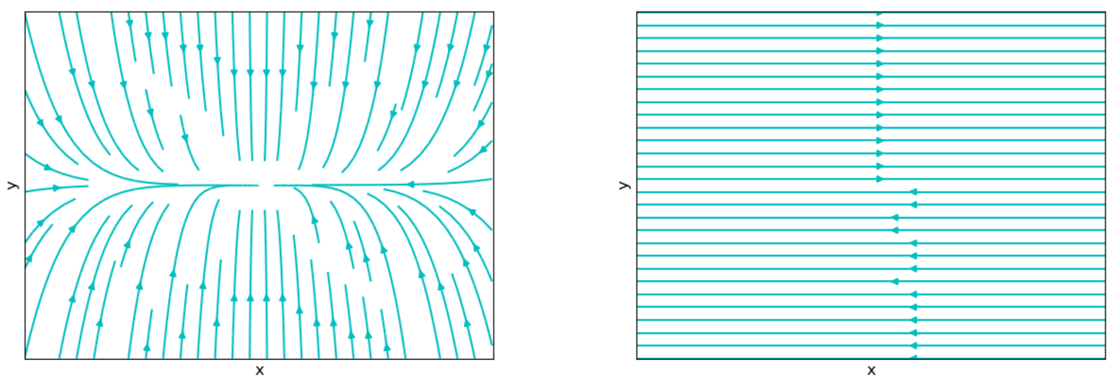

Figure 21: (Left) Improper Nodal Source - $\lambda_1 < \lambda_2 < 0$. (Right) Double Zero Eigenvalue (one eigenvector) $\lambda_1 = \lambda_2 = 0$.

4.3 Fundamental matrix solution:

The solution vectors of an $n \times n$ homogeneous linear system

$$\mathbf{x}' = A\mathbf{x}, \tag{4.10}$$

can be used to construct a square matrix $\mathbf{X} = \mathbf{\Phi}(t)$ that satisfies the *matrix differential equation*

$$\mathbf{X}' = A\mathbf{X}, \tag{4.11}$$

associated with (4.10). Suppose that $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are n linearly independent solutions of (4.10). Then the $n \times n$ matrix

$$\mathbf{\Phi}(t) = \begin{pmatrix} | & & | \\ \mathbf{x}_1(t) & \ldots & \mathbf{x}_n(t) \\ | & & | \end{pmatrix}, \tag{4.12}$$

having these solution vectors as its column vectors, is called a **fundamental matrix** for the system in (4.10).

Since the column vector $\mathbf{x}(t) = \mathbf{x}_j(t)$ of the fundamental matrix $\mathbf{\Phi}(t)$ in (4.12) satisfies the differential equation $\mathbf{x}' = A\mathbf{x}$, it follows (from the definition of matrix multiplication) that the matrix $\mathbf{X} = \mathbf{\Phi}(t)$ itself satisfies the matrix differential equation $\mathbf{X}' = A\mathbf{X}$. Further, because its column vectors are linearly independent, it also follows that the fundamental matrix $\mathbf{\Phi}(t)$ is nonsingular, and therefore has an inverse matrix $\mathbf{\Phi}(t)^{-1}$.

Conversely, any nonsingular matrix solution $\mathbf{\Psi}(t)$ of (4.11) has linearly independent column vectors that satisfy (4.10), so $\mathbf{\Psi}(t)$ if a fundamental matrix for the system (4.10).

Recall that a general solution of (4.10) is a linear combination of n linearly independent solutions, i.e.,

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t). \tag{4.13}$$

In terms of the fundamental matrix (4.12) we can write Eq. (4.13) as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}, \tag{4.14}$$

where $\mathbf{c} = (c_1, \ldots, c_n)^T$ is an arbitrary *constant* vector. It follows that if $\mathbf{\Psi}(t)$ is any other fundamental matrix of (4.10), then each column vector of $\mathbf{\Psi}(t)$ is a linear combination of the column vectors of $\mathbf{\Phi}(t)$. From Eq. (4.14) we get

$$\mathbf{\Psi}(t) = \mathbf{\Phi}(t)\mathbf{C}, \tag{4.15}$$

for some $n \times n$ matrix $\mathbf{C}$ of constants.

Next, suppose that (4.10) is given the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Then it is easy to see:

$$\begin{aligned}\mathbf{x}_0 &= \mathbf{x}(0)\\ &= \mathbf{\Phi}(0)\mathbf{c}\end{aligned}$$

which gives

$$\mathbf{c} = \mathbf{\Phi}(0)^{-1}\mathbf{x}_0.$$

Remark 8: Since matrix multiplication is not commutative we must make sure to multiply $\mathbf{x}_0$ by $\mathbf{\Phi}(0)^{-1}$ on the *left*.

Theorem 4.4. Fundamental matrix solutions
Let $\mathbf{\Phi}(t)$ be a fundamental matrix solution for the homogeneous linear system $\mathbf{x}' = A\mathbf{x}$. Then the unique solution of the initial-value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.16}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0 \tag{4.17}$$

Remark 9: An advantage of the fundamental matrix approach is this: Once we know the fundamental matrix $\mathbf{\Phi}(t)$ and the inverse matrix $\mathbf{\Phi}(0)^{-1}$, we can calculate rapidly by matrix multiplication the solutions corresponding to different initial conditions.

It turns out that we can calculate a fundamental matrix solution directly from the matrix A. Recall the first-order initial-value problem $x' = ax$, $x(0) = x_0$ has solution $x(t) = x_0 e^{at}$. It turns out that we can define a concept of 'matrix exponential' in such a way that

$$\mathbf{X}(t) = \mathrm{e}^{At},$$

is a matrix solution of the matrix differential equation

$$\mathbf{X}' = A\mathbf{X},$$

with $n \times n$ coefficient matrix A.

Recall that the exponential e^z of the complex number z may be defined by means of the exponential series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \ldots. \tag{4.18}$$

Similarly, if A is an $n \times n$ matrix, then the **exponential matrix $\mathbf{e^A}$** is the $n \times n$ matrix defined by the series

$$\mathrm{e}^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} = \mathbb{I} + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \ldots, \tag{4.19}$$

where $\mathbb{I}$ is the identity matrix. It can be shown that the exponential matrix e^A is defined (by (4.19)) for every square matrix A. That is,

$$\lim_{k\to\infty} \left(\sum_{n=0}^{k} \frac{A^n}{n!} \right)$$

exists for any square matrix A.

Remark 10: If $A^n = 0$ for some positive integer n then the series (4.19) terminates after a finite number of terms and the matrix exponential e^A is readily calculated. A matrix with this property is called **nilpotent**.

Remark 11: If A is **diagonal** then the exponential matrix e^A is the exponential of each term along the main diagonal of A.

Example 4.4. Suppose that

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

then it follows

$$\mathrm{e}^A = \begin{pmatrix} e^{-2} & 0 & 0 \\ 0 & e^7 & 0 \\ 0 & 0 & e^4 \end{pmatrix}.$$

Theorem 4.5. Matrix exponential solutions
If A is an $n \times n$ matrix, then the solution of the initial-value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{4.20}$$

is given by

$$\mathbf{x}(t) = \mathrm{e}^{At}\mathbf{x}_0, \tag{4.21}$$

and this solution is unique.

It also follows from the above theorem that $\mathrm{e}^{At} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$. Thus we can find the matrix exponential by solving the linear system (4.10).

4.4 Variation of parameters:

We now turn our attention to nonhomogeneous linear systems. Recall the nonhomogeneous linear system (4.7) can be written in the concise form

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t), \tag{4.22}$$

where $\mathbf{x} = (x_1, \dots, x_n)^T$, $P(t)$ is $n \times n$ matrix-valued function, and $\mathbf{f} = (f_1, \dots, f_n)^T$.

It follows that the general solution of the nonhomogeneous system (4.22) is given by

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t), \tag{4.23}$$

where $\mathbf{x}_c$ is the solution to the associated homogeneous system and $\mathbf{x}_p$ is any particular solution to (4.7). Thus we discuss here a method to generate a particular solution to the nonhomogeneous linear system (4.22).

Suppose that $\mathbf{x}_c(t) = \mathbf{\Phi}(t)\mathbf{c}$ is the complementary solution to the associated homogeneous of (4.22). Now we must find a particular solution. Consider the following:

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\mathbf{u}(t). \tag{4.24}$$

Taking a derivative of (4.24) gives

$$\mathbf{x}_p'(t) = \mathbf{\Phi}'(t)\mathbf{u}(t) + \mathbf{\Phi}(t)\mathbf{u}'(t).$$

Since $\mathbf{x}_p$ is a particular solution of (4.22) we also have

$$\mathbf{x}_p'(t) = P(t)\mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{f}(t).$$

Combining these two results yields

$$\mathbf{\Phi}'(t)\mathbf{u}(t) + \mathbf{\Phi}(t)\mathbf{u}'(t) = P(t)\mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{f}(t).$$

Finally since $\mathbf{\Phi}(t)$ is a fundamental matrix it follows that $\mathbf{\Phi}'(t) = P(t)\mathbf{\Phi}(t)$ and thus

$$\mathbf{u}'(t) = \mathbf{\Phi}^{-1}(t)\mathbf{f}(t).$$

This leads to the following theorem.

Theorem 4.6. Variation of parameters
If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}'(t) = P(t)\mathbf{x}$ on some interval where $P(t)$ and $\mathbf{f}(t)$ are continuous, then a particular solution of the nonhomogeneous system

$$\mathbf{x}'(t) = P(t)\mathbf{x} + \mathbf{f}(t) \tag{4.25}$$

is given by

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int_a^t \mathbf{\Phi}^{-1}(s)\mathbf{f}(s)ds. \tag{4.26}$$

Remark 12: Note that we must be able to first find a fundamental matrix solution to the associated nonhomogeneous system in order to apply the method. Also, integration is done componentwise.

Thus, a particular solution to the nonhomogeneous initial-value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(0)\mathbf{x}_0 + \mathbf{\Phi}(t) \int_0^t \mathbf{\Phi}^{-1}(s)\mathbf{f}(s)ds. \tag{4.27}$$

5 Nonlinear systems of differential equations:

We now examine autonomous nonlinear systems of differential equations.

$$\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1, \ldots, x_n) \\
\frac{dx_2}{dt} &= F_2(x_1, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= F_n(x_1, \ldots, x_n)
\end{aligned} \tag{5.1}$$

The above system can be written using the compact notation

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}), \tag{5.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field given by

$$\mathbf{F}(\mathbf{x}) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))^T. \tag{5.3}$$

For convenience we will restrict our attention to nonlinear systems in the phase plane, i.e.,

$$\begin{aligned}
\frac{dx}{dt} &= F(x, y) \\
\frac{dy}{dt} &= G(x, y)
\end{aligned} \tag{5.4}$$

but it should be noted that many of the results generalize to higher dimensions.

We generally assume that the functions F and G are continuously differentiable in some region R of the xy-plane. Then according to the existence and uniqueness theorems, given t_0 and any point (x_0, y_0) of R, there is a *unique* solution $x = x(t)$, $y = y(t)$ of (5.4) that is defined on some open interval (a, b) containing t_0 and satisfies the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$. The equations $x = x(t)$, $y = y(t)$ then describe a parametrized solution curve in the phase plane. Any such solution is called a **trajectory** of the system (5.4), and precisely one trajectory passes through each point of the region R.

A **critical point** of the system (5.4) is a point (x^*, y^*) such that

$$F(x^*, y^*) = G(x^*, y^*) = 0.$$

Note that at the critical point (x^*, y^*) we have $x' = y' = 0$. Thus, $x(t) \equiv x^*$ and $y(t) \equiv y^*$ is a constant solution which we refer to as an **equilibrium solution**. In many applications these solutions are of greatest interest.

5.1 Phase portraits:

If the initial point (x_0, y_0) is not a critical point, then the corresponding trajectory is a curve in the xy-plane along which the point $(x(t), y(t))$ moves as t increases. It turns out that any trajectory not consisting of a single point is a nondegenerate curve with no self-intersections. We can exhibit qualitatively the behavior of solutions of the autonomous system (5.4) by constructing a picture that shows its critical points together with a collection of of typical solution curves or trajectories in the xy-plane. Such a picture is called a **phase portrait** because it illustrates 'phases' or xy-states of the system, and indicates how they change with time.

Another way to visualize the system is to construct a **slope field** in the xy-phase plane by drawing line segments having slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x,y)}{F(x,y)}, \tag{5.5}$$

where (5.5) is the slope of a parametric curve in the plane. Finally, one could draw a **direction field** by drawing typical vectors pointing in the direction motion at each point (x, y) given by $(F(x, y), G(x, y))^T$.

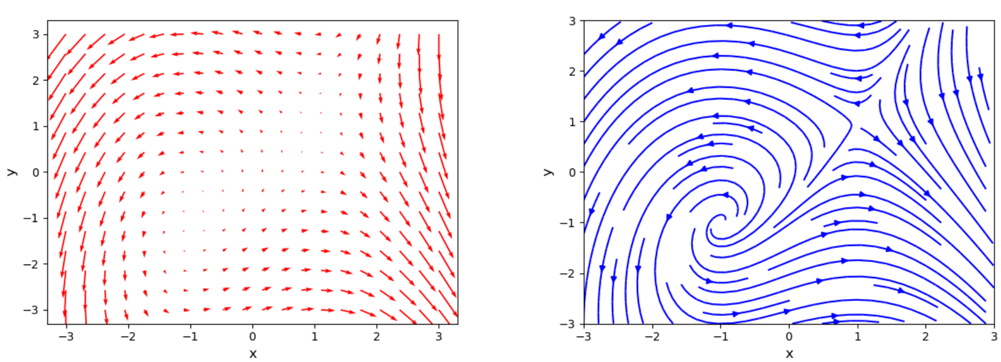

Figure 22: (Left) Direction field of the system $x' = x - y$ and $y' = 1 - x^2$.
(Right) Phase portrait of the system $x' = x - y$ and $y' = 1 - x^2$.

5.2 Stability:

A critical point $\mathbf{x}^* = (x^*, y^*)$ of the autonomous system (5.4) is **stable** provided for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\mathbf{x}_0 - \mathbf{x}^*| < \delta \implies |\mathbf{x}(t) - \mathbf{x}^*| < \epsilon \tag{5.6}$$

for all $t > 0$. This means that if the initial condition is sufficiently close to the critical point then the solution remains close to the critical point for all time.

Note that (5.6) holds if $\mathbf{x}(t) \to \mathbf{x}^*$, as $t \to \infty$. Thus the nodal sinks discussed in the section on linear systems can also be described as *stable nodes*. A critical point (x^*, y^*) that is not stable is called **unstable**.

The critical point (x^*, y^*) is called **asymptotically stable** if there exists a $\delta > 0$ such that

$$|\mathbf{x} - \mathbf{x}^*| < \delta \implies \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*. \tag{5.7}$$

Thus, a spiral sink is asymptotically stable while a center is simply stable.

5.3 Polar coordinates:

Sometimes converting to polar coordinates can help one analyze the dynamics of a nonlinear system.

Example 5.1. Consider the system of differential equation given by

$$\begin{aligned} x' &= -ky + x(1 - x^2 - y^2) \\ y' &= kx + y(1 - x^2 - y^2) \end{aligned}$$

where k is a constant. It is not hard to show that the only critical point of the system is given by $(0, 0)$. Further it turns out that one can solve this nonlinear system exactly by converting to polar coordinates.

$$x(t) = r(t)\cos\theta(t)$$

$$y(t) = r(t)\sin\theta(t)$$

$$r(t)^2 = x(t)^2 + y(t)^2$$

$$\theta(t) = \tan^{-1}\left(\frac{y(t)}{x(t)}\right)$$

Thus,

$$\theta'(t) = \frac{1}{1 + (y/x)^2}\left(\frac{y}{x}\right)' = \left(\frac{x^2}{x^2 + y^2}\right)\left(\frac{xy' - yx'}{x^2}\right) = \frac{xy' - yx'}{x^2 + y^2}$$

and substitution into the system yields

$$\theta'(t) = \frac{k(x^2 + y^2)}{x^2 + y^2} = k$$

which can be integrated to yield $\theta(t) = kt + \theta_0$. So, $\theta \to \infty$ as $t \to \infty$. Next,

$$2rr'(t) = 2xx' + 2yy'$$

and substitution into the above system yields

$$2rr'(t) = (x^2 + y^2)(1 - x^2 - y^2) = r(1 - r^2)$$

which can be integrated using partial fraction to yield $r(t) = \frac{r_0}{\sqrt{r_0^2 + (1 - r_0^2)e^{-2t}}}$. So, $r \to 1$ as $t \to \infty$. Taking both equations into consideration we see that for $r_0 > 1$ the trajectories spiral in towards the circle $r \equiv 1$. Also, for $0 < r_0 < 1$ the trajectories spiral out towards the circle $r \equiv 1$. Further all motion is counter-clockwise and the origin is an unstable critical point. Let's confirm all of this by plotting the trajectories numerically. Thus, from Fig. 23 we see that indeed all trajectories spiral towards the curve $r \equiv 1$. The curve

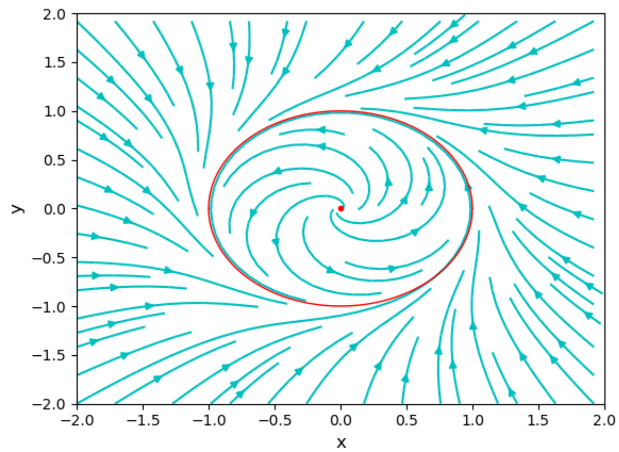

Figure 23: Phase portrait for the system $x' = -y + x(1 - x^2 - y^2)$ and $y' = x + y(1 - x^2 - y^2)$. The circle $r \equiv 1$ (red curve) is a limit cycle and the origin is an unstable critical point (red dot).

$r \equiv 1$ is called a **limit cycle**.

Under rather general hypotheses it can be shown that there are four possibilities for a nondegenerate trajectory of the autonomous system

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y).$$

The four possibilities are:

1. $(x(t), y(t))$ approaches a critical point as $t \to \infty$.
2. $(x(t), y(t))$ is unbounded with increasing t.
3. $(x(t), y(t))$ is a periodic solution with closed trajectory.
4. $(x(t), y(t))$ spirals toward a closed trajectory as $t \to \infty$.

As a consequence, the qualitative nature of the phase plane portrait of the trajectories of an autonomous system is determined largely by the locations of its critical points and by the behavior of its trajectories near its critical points.

5.4 Linearization:

In this section we will discuss how to linearize a nonlinear system about a critical point. Then we will use the linearized system to examine the nonlinear dynamics near critical points. An important question is whether or not the linearization is a good approximation of the nonlinear system near critical points.

Suppose f is defined on an open interval I and differentiable at a point $x^* \in I$. Then from calculus we know that the curve f can be locally approximated by a line whose slope is $f'(x^*)$. Specifically,

$$f(x^* + \Delta x) = f(x^*) + f'(x^*)\Delta x + o(\Delta x)$$

as $\Delta x \to 0$. It turns out that linear approximation can be generalized to vector fields. Consider the nonlinear

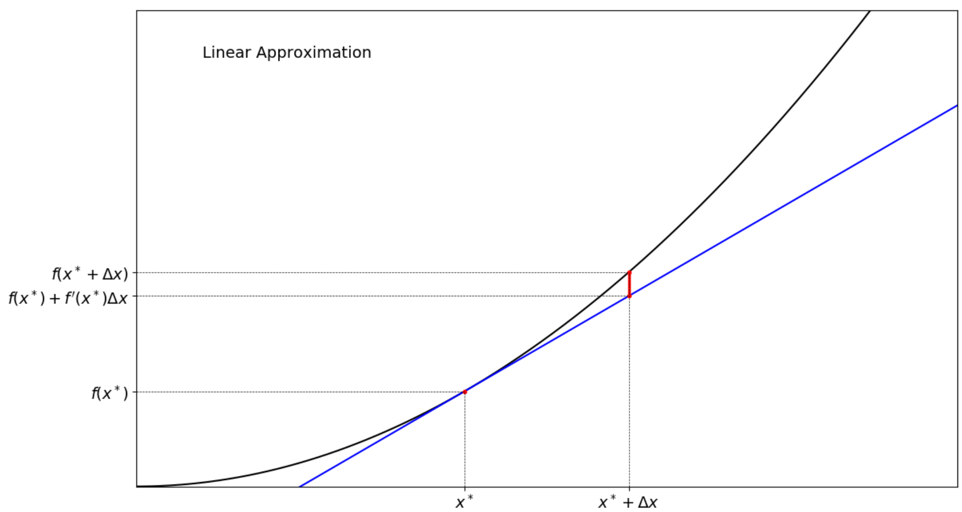

Figure 24: Linear approximation of a curve near $x = x^*$.

system of differential equations given by

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}). \tag{5.8}$$

Suppose that $x = x^*$ is a critical point of (5.8) and consider a small perturbation, i.e., $\mathbf{x}^* + \mathbf{\Delta x} = (x_1 + \Delta x_1, \ldots, x_n + \Delta x_n)^T$. Then we have:

$$\begin{aligned} \mathbf{x}' &= \mathbf{F}(\mathbf{x}) \\ (\mathbf{x}^* + \mathbf{\Delta x})' &= \mathbf{F}(\mathbf{x}^* + \mathbf{\Delta x}) \\ &= \mathbf{F}(\mathbf{x}^*) + D\mathbf{F}(\mathbf{x}^*)\mathbf{\Delta x} + o(|\mathbf{\Delta x}|), \quad \text{as } \mathbf{\Delta x} \to \mathbf{0} \\ &= D\mathbf{F}(\mathbf{x}^*)\mathbf{\Delta x} + o(|\mathbf{\Delta x}|), \quad \text{as } \mathbf{\Delta x} \to \mathbf{0} \end{aligned}$$

Thus, the linearization of a nonlinear system about the critical point $\mathbf{x} = \mathbf{x}^*$ is given by

$$(\mathbf{\Delta x})' = D\mathbf{F}(\mathbf{x}^*)\mathbf{\Delta x},$$

where $D\mathbf{F}(\mathbf{x}^*)$ is the Jacobian matrix evaluated at the critical point. This is just a linear system with constant coefficients! Also, we define the Jacobian matrix as follows:

$$D\mathbf{F}(\mathbf{x}) := \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}, \tag{5.9}$$

which is just a matrix of partial derivatives of the component functions of $\mathbf{F}$.

Next we answer when the linearization about a critical point is a good approximation of the nonlinear dynamics.

Theorem 5.1. Stability of linear systems
Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix A of the two-dimensional linear system

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{5.10}$$

with $ad - bc \neq 0$. Then the critical point $(0, 0)$ is

1. Asymptotically stable if the real parts of λ_1 and λ_2 are both negative;
2. Stable but not asymptotically stable if the real parts of λ_1 and λ_2 are both zero (so that $\lambda_1, \lambda_2 = \pm iq$);
3. Unstable if either λ_1 or λ_2 has a positive real part.

Theorem 5.2. Stability of linearized systems
Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix of the linear system

$$(\mathbf{\Delta x})' = D\mathbf{F}(\mathbf{x}^*)\mathbf{\Delta x}$$

associated with the nonlinear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. Then

1. If $\lambda_1 = \lambda_2$ are equal real eigenvalues, then the critical point $\mathbf{x}^*$ of the linearized system is either a node or a spiral point, and is asymptotically stable if $\lambda_1 = \lambda_2 < 0$, unstable if $\lambda_1 = \lambda_2 > 0$.
2. If λ_1 and λ_2 are pure imaginary, then $\mathbf{x}^*$ is either a center or a spiral point, and may be either asymptotically stable, stable, or unstable.
3. Otherwise – that is, unless λ_1 and λ_2 are either real equal or pure imaginary – the critical point $\mathbf{x}^*$ of the nonlinear system is of the same type and stability as the critical point $(0, 0)$ of the associated linear system.

An important consequence of the classification of cases in Theorem 5.2 is that a *critical point of a linearizable system is asymptotically stable if it is an asymptotically stable critical point of the linearization of the system*. Moreover, a critical point of the linearizable system is unstable if it is an unstable critical point of the linearized system. Thus, the linearization provides lots of information regarding the nonlinear dynamics.

Example 5.2. Consider the nonlinear system given by

$$x' = x - y^2,$$
$$y' = x - y.$$

Classify the dynamics of this system near the critical points.

Step 1: Find all critical points.

We must solve the system of equations:

$$x - y^2 = 0$$
$$x - y = 0 \iff x = y$$

So, plugging into the first equation $y - y^2 = 0 \iff y(y-1) = 0 \iff y = 0$ or $y = 1$. Thus, the critical points are given by $(0,0)$ and $(1,1)$.

Step 2: Find the Jacobian matrix.

$$D\mathbf{F}(\mathbf{x}) = \begin{pmatrix} 1 & -2y \\ 1 & -1 \end{pmatrix}$$

Step 3: Evaluate the Jacobian matrix at each critical point and find the eigenvalues.

$$|D\mathbf{F}(0,0) - \lambda\mathbb{I}| = 0 \iff -(1-\lambda)(1+\lambda) = 0 \iff \lambda_{1,2} = \pm 1$$

.

$$|D\mathbf{F}(1,1) - \lambda\mathbb{I}| = 0 \iff \lambda^2 + 1 = 0 \iff \lambda_{1,2} = \pm i$$

Step 4: Apply Theorem 5.2 to classify the trajectories of the system.

The critical point $(0,0)$ is an unstable saddle of the nonlinear system. We see this since the eigenvalues of the linearized system are real and of opposite sign.

On the other hand the critical point $(1,1)$ might be asymptotically stable spiral, stable center, or unstable spiral since the eigenvalues of the linearized system are purely imaginary. Thus in this case Theorem 5.2 is inconclusive.

Let's plot the trajectories to classify the critical point $(1,1)$.

6 Some special equations:

Second-order Euler equation:

$$ax^2y'' + bxy' + cy = 0 \tag{6.1}$$

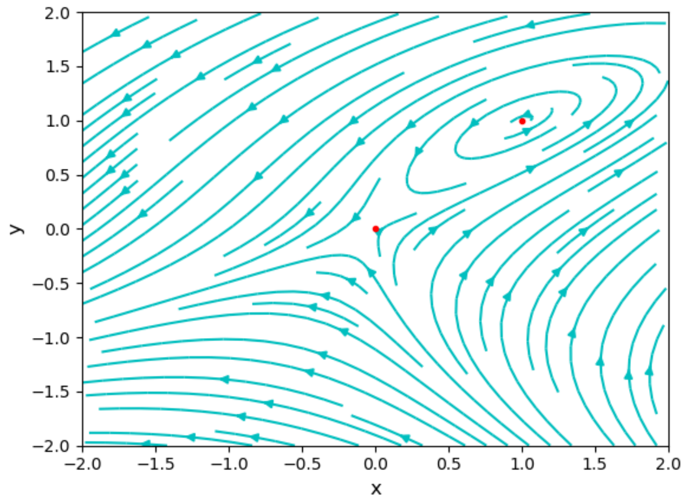

Figure 25: Nonlinear system $x' = x - y^2$ and $y' = x - y$. Critical point $(0, 0)$ is unstable saddle. Critical point $(1, 1)$ is a stable center.

Consider the invertible substitution $v = \ln x$, $x > 0$. Since $y = y(v(x))$ the chain rule gives:

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dv^2}\Big(\frac{dv}{dx}\Big)^2 + \frac{dy}{dv}\frac{d^2v}{dx^2} = \Big(\frac{d^2y}{dv^2} + \frac{dy}{dv}\Big)\frac{1}{x^2}$$

and substituting into (6.1) we have

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0. \tag{6.2}$$

Thus, we've transformed the second-order Euler equation (6.1) into the second-order equation with constant coefficients (6.2). Thus, by solving (6.2) and using $v = \ln x$ we can also solve (6.1). As expected the solution is dependent on the roots of the characteristic equation.

Case (i): Distinct real roots ($r_1 \neq r_2$).

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2} \tag{6.3}$$

Case (ii): Repeated real root ($r_{1,2} = r$).

$$y(x) = (c_1 + c_2 \ln x)x^r \tag{6.4}$$

Case (iii): Complex conjugate pair of roots ($r_{1,2} = \alpha \pm i\beta$).

$$y(x) = x^\alpha(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)) \tag{6.5}$$

Reduction of order:

Consider the homogeneous linear differential equation:

$$y'' + p(x)y' + q(x)y = 0, \tag{6.6}$$

where p and q are continuous functions on some open interval I. We know that a general solution of (6.6) is given by a linear combination of two linearly independent solutions. Suppose we know one solution of (6.6) is given by $y_1 = y_1(x)$. Consider a second solution of the form

$$y_2(x) = v(x)y_1(x). \tag{6.7}$$

If we can find $v = v(x)$ then we have a second linearly independent solution of (6.6) and thus have solved the problem. Plugging (6.7) into (6.6) and simplifying gives

$$y_1 v'' + (2y_1' + py_1)v' = 0. \tag{6.8}$$

Letting $u = v'$ gives

$$y_1 u' + (2y_1' + py_1)u = 0, \tag{6.9}$$

which is a linear first-order equation in $u = u(x)$. Thus, one can find u and then integrate it to find v. Hence one really only needs to find a single solution to a second-order linear differential equation. Once a solution is found a second can be easily constructed.

Example 6.1. Consider Legendre's equation of order 1:

$$(1 - x^2)y'' - 2xy' + 2y = 0 \tag{6.10}$$

It is not hard to see that $y_1(x) = x$ is one solution to this equation. To find a second independent solution we apply reduction of order. Let $y_2(x) = v(x)y_1(x) = vx$ be a second solution. Then

$$xu' + \Big(2 - \frac{2x^2}{1-x^2}\Big)u = 0 \iff u' = -\Big(\frac{2}{x} - \frac{2x}{1-x^2}\Big)u$$

where $u = v'$ and $p(x) = -2x/(1-x^2)$. Thus,

$$u(x) = \frac{C}{x^2(1+x)(1-x)},$$

and applying partial fractions gives

$$v(x) = \frac{1}{x} - \frac{1}{2}\frac{\ln(1+x)}{\ln(1-x)}.$$

Thus, we have constructed a second solution to Legendre's equation of order 1, namely,

$$y_2(x) = 1 - \frac{x}{2}\frac{\ln(1+x)}{\ln(1-x)}.$$

Time-independent Schrödinger equation:

Consider the following expression:

$$\big(-\partial_x^2 + V(x)\big)\psi = \lambda\psi, \tag{6.11}$$

where $V = V(x)$ is a potential function and λ is the total energy in the system. By defining the differential operator $\mathcal{L} := \big(-\partial_x^2 + V(x)\big)$ we can write (6.11) as

$$\mathcal{L}\psi = \lambda\psi \tag{6.12}$$

which is an eigenvalue problem. Instead of finding the eigenvalues of a matrix though we are now looking for the eigenvalues of a linear operator. Finally, we can also rewrite (6.11) in the form

$$\psi'' + Q(x)\psi = 0, \tag{6.13}$$

where $Q(x) = \lambda - V(x)$. Despite looking harmless, closed-form solutions of (6.13) have only been found in a few special cases. One way to analyze the time-independent Schrödinder equation is by using asymptotic approximations such as the famous Wentzel-Kramers-Brillouin (WKB) method. Values of x where $Q(x) = 0$ are called 'turning points' since at these points $\lambda = V(x)$ and so there is no kinetic energy in the system.

Riccati equation:

A Riccati equation is a first-order nonlinear differential equation of the form:

$$y' = A(x)y^2 + B(x)y + C(x) \tag{6.14}$$

Suppose that one particular solution $y_1 = y_1(x)$ of (6.14) is known. Then making the substitution

$$y(x) = y_1(x) + \frac{1}{v(x)}$$

transforms the Riccati equation into the *linear* equation

$$v' + (B + 2Ay_1)v = -A$$

which is readily solvable. Thus, once we know any particular solution of a Riccati equation we can find a general solution by applying the above transformation. Note that the Riccati equation is nonlinear and thus there may exist singular solutions as well.

Clairaut equation:

A Clairaut equation is a first-order equation of the form:

$$y = xy' + g(y') \tag{6.15}$$

It is easy to see that $y(x) = Cx + g(C)$, where C a constant is a general solution of (6.15).

Bessel's equation of order n:

Bessel's equation of order n is a second-order linear differential equation of the form:

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{6.16}$$

Bessel's equation appears in many applications such as acoustics, heat flow and electromagnetic radiation.

Legendre's equation of order n:

Legendre's equation of order n is a second-order linear differential equation of the form:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{6.17}$$

Legendre's equation appears in spherical harmonics.

Airy's equation:

Airy's equation is a second-order linear differential equation of the form:

$$y'' = xy \tag{6.18}$$

Notice that this is just the time-independent Schrödinger equation with $Q(x) = -x$. This is the simplest second-order linear differential equation with a turning point.

7 Power series methods:

In this section we will use power series to derive representations of solutions to linear differential equations with variable coefficients. We will restrict attention to second-order equations, but the technique can be generalized to higher order equations. Recall a second-order linear differential equation is given by:

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{7.1}$$

7.1 Power series:

To analyze (7.1) we will use power series. Recall that a power series **centered at** $x = a$ is given by an infinite polynomial expansion, i.e.,

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \tag{7.2}$$

and n is an integer. Since (7.2) is an infinite series we must be careful to check that it converges. The power series converges on the interval I provided that the limit

$$\sum_{n=0}^{\infty} c_n(x-a)^n = \lim_{N\to\infty} \sum_{n=0}^{N} c_n(x-a)^n$$

exists for all $x \in I$.

Theorem 7.1. Radius of convergence
Given the power series $\sum c_n(x-a)^n$, suppose that the limit

$$\rho = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \tag{7.3}$$

exists (ρ is finite) or is infinite (in this case we will write $\rho = \infty$). Then

a. If $\rho = \infty$, then the series diverges for all $x \neq a$.

b. If $0 < \rho < \infty$, then $\sum c_n x^n$ converges if $|x - a| < \rho$ and diverges if $|x - a| > \rho$.

c. If $\rho = \infty$, then the series converges for all x.

Thus, if the power series $\sum c_n (x-a)^n$ converges on the open interval I then it defines a function and we can write $y(x) = \sum c_n (x-a)^n$. In this case we call the series $\sum c_n (x-a)^n$ the **power series representation** of the function $y = y(x)$.

Here are some examples of power series representations of well-known functions:

(i.) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(ii.) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

(iii.) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1} x^{2n+1}}{(2n+1)!}$

(iv.) $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

(v.) $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

(vi.) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad |x| < 1$

(vii.) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$

(viii.) $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \ldots, \quad |x| < 1$

Remark 1: Power series such as those listed above are often derived as Taylor series. The **Taylor series** with **center** at $x = a$ of the function f is the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots \tag{7.4}$$

Notice that by truncating a Taylor series at $n = N$ we get a local approximation of the function f near the center $x = a$.

Remark 2: If the Taylor series of a function $f = f(x)$ converges for all x in some open interval I containing a, then we say that the function f is **analytic** at $x = a$. For example,

– every polynomial is analytic everywhere;

– every rational function is analytic wherever its denominator is nonzero;

– more generally, if the two functions f and g are both analytic at $x = a$, then so are their sum $f + g$ and their product $f \cdot g$, as is their quotient f/g if $g(a) \neq 0$.

Fortunately, power series can be manipulated algebraically in much the same way as polynomials. For example, if $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$, then

$$f(x) + g(x) = \sum (a_n + b_n)x^n$$

and

$$f(x)g(x) = \sum c_n x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

Sometimes the best way to derive a power series of a function is by using a known a power series.

Example 7.1. Find a power series representation of the function

$$f(x) = \frac{1}{2 + x^2}.$$

What is the radius of convergence? Rewrite the above function as follows:

$$\frac{1}{2+x^2} = \frac{1}{2(1-(-x^2/2))} = \frac{1}{2}\Big(\frac{1}{1-(-x^2/2)}\Big) = \frac{1}{2}\Big(\frac{1}{1-u}\Big) = \frac{1}{2}\sum_{n=0}^{\infty} u^n, \qquad |u| < 1$$

where $u = -x^2/2$. Thus,

$$\frac{1}{2+x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n}, \qquad |x| < \sqrt{2}.$$

Theorem 7.2. Termwise differentiation of power series
If the power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \tag{7.5}$$

of the function f converges on the open interval I, then f is differentiable on I, and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots \tag{7.6}$$

at each point of I.

The next theorem is especially important for solving differential equations using power series.

Theorem 7.3. Identity principle
If

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

for every point x in some open interval I, then $c_n = d_n$ for an $n \geq 0$.

Next we discuss **shift of index of summation** which will be relevant to solving differential equations using power series. Simply put if we increase the index of summation by k then we must decrease the starting point by k as well. Likewise if we decrease the index of summation by j then we must increase the starting point by j as well. For example,

$$\sum_{n=2}^{\infty}(n-1)c_{n-1}x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^{n+1},$$

so by decreasing the starting point by 2 we must increase the index by 2 to maintain equality.

7.2 Power series method:

Now we discuss how to solve a differential equation of the form (7.1) using power series. Essentially the technique is analogous to the method of undetermined coefficients.

Step 1: Assume a solution in the form of a power series, i.e., $y(x) = \sum_{n=0}^{\infty} c_n x^n$.

Step 2: Use termwise differentiation to find y' and y''. Then substitute into the differential equation and simplify.

Step 3: Shift the index of summation so that all the terms balance. Then use the identity principle to find a **recurrence relation** for the coefficients c_n.

Step 4: Solve the **recurrence relation** to get a formula for c_n, $n \geq 0$ thus solving the differential equation.

Example 7.2. Find a general solution of the differential equation $y' - xy = 0$. We begin by assuming $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$. This gives,

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0 \iff \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

Next, to balance terms we first shift the index of summation in the second term down two units (and thus shift the starting point up two units). Thus, we can rewrite as follows:

$$c_1 + \sum_{n=2}^{\infty} nc_n x^{n-1} - \sum_{n=2}^{\infty} c_{n-2} x^{n-1} = 0 \iff c_1 + \sum_{n=2}^{\infty}[nc_n - c_{n-2}]x^{n-1} = 0.$$

Thus, c_0 is arbitrary, $c_1 = 0$, and

$$c_n = \frac{c_{n-2}}{n}.$$

Finally, to solve the recurrence relation we calculate the first few terms.

$$c_2 = \frac{c_0}{2}, \qquad c_3 = 0, \qquad c_4 = \frac{c_2}{4} = \frac{c_0}{2 \cdot 4}, \qquad c_5 = 0, \qquad c_6 = \frac{c_4}{6} = \frac{c_0}{2 \cdot 4 \cdot 6}.$$

Thus, $c_{2n} = \frac{c_0}{2^n n!}$ and $c_{2n+1} = 0$ for $n \geq 0$. Thus a power series solution is given by

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = c_0 e^{x^2/2}.$$

7.3 Ordinary and singular points:

The next logical question to answer is when does a power series solution of (7.1) exist? Rewrite (7.1) in the standard form:

$$y'' + P(x)y' + Q(x)y = 0 \tag{7.7}$$

where $P = B/A$ and $Q = C/A$. Assume that A, B and C are analytic at $x = a$. The point $x = a$ is called an **ordinary point** of (7.7) – and of the equivalent (7.1) – provided that the functions $P = P(x)$ and $Q = Q(x)$ are both analytic at $x = a$. Otherwise, $x = a$ is a **singular point**. It follows that, if $A(a) \neq 0$ in (7.1) with analytic coefficients, the $x = a$ is an ordinary point. If A, B, and C are *polynomials* with no common factors, then $x = a$ is an ordinary point if and only if $A(a) \neq 0$.

Example 7.3. Find all ordinary points of the differential equation:

$$xy'' + (\sin x)y' + x^2 y = 0$$

Begin by rewriting the differential equation in the standard form, i.e.,

$$y'' + \left(\frac{\sin x}{x}\right)y' + xy = 0$$

and note that the only possible singular point is at $x = 0$. Other wise $P(x) = \sin x/x$ is a quotient of analytic functions. Thus, to examine whether $x = 0$ is an ordinary point we use the power series representation of $\sin x$. Thus,

$$P(x) = \frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

and so $P(x)$ has a convergent power series representation. Thus $x = 0$ is an ordinary point.

Theorem 7.4. Solutions near ordinary points
Suppose that a is an ordinary point of the equation

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{7.8}$$

that is, the functions $P = B/A$ and $Q = C/A$ are analytic at $x = a$. Then (7.8) has two linearly independent solutions, each of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{7.9}$$

The radius of convergence of any such series solution is *at least* as large as the distance from a to the nearest (real or complex) singular point of (7.8). The coefficients in the series (7.9) can be determined by its substitution in (7.8).

Remark 3: A general solution to a linear second-order differential equation is given by a linear combination of two linearly independent solutions. One can use the simple ICs

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0 \qquad\qquad y(0) = 0 \quad \text{and} \quad y'(0) = 1$$

to construct two linearly independent solutions from the power series solution.

Remark 4: The center of the power series a is determined by the initial condition, i.e., if $y(x_0) = y_0$ is the initial condition then our power series solution should be of the form $y(x) = \sum c_n(x - x_0)^n$.

Remark 5: A problem with a power series solution of the form $\sum c_n(x - a)^n$ can always be translated to a problem with a power series solution of the form $\sum c_n t^n$ by making the substitution $t = x - a$. One can also translate a singular point at $x = a$ to a singular point at $t = 0$ using the same substitution. So, without loss of generality we may restrict our attention to case in which $x = 0$ is a singular point of (7.8).

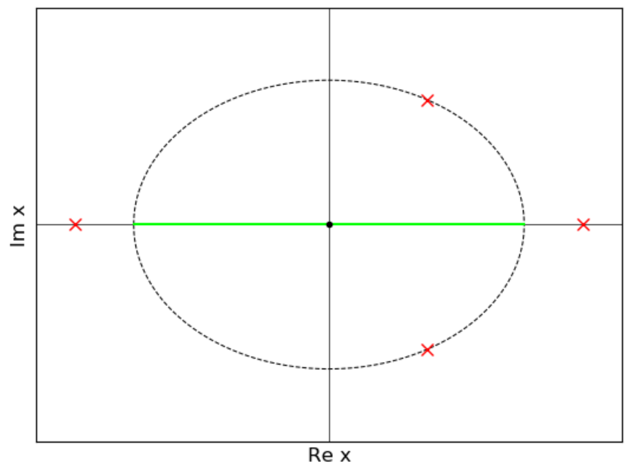

Figure 26: Graphical depiction of Theorem 7.4 where the singular points are depicted by $\times$ (red). The power series solution exists on an open interval about the origin (lime).

Next, if $x = 0$ is a singular point then Theorem 7.4 does not hold and so power series solutions may not exist. In order to analyze the possible form of solutions in this case we take a closer look at singular points. Recall that $x = 0$ is an ordinary point (as opposed to a singular point) of (7.8) if the functions P and Q are analytic at $x = 0$; that is, if P and Q have convergent power series expansions in powers of x on some open interval containing $x = 0$. Now it can be proved that each of the functions P and Q *either* is analytic *or* approaches $\pm\infty$ as $x \to 0$. Consequently, $x = 0$ is a singular point of (7.8) provided that either P or Q (or both) approaches $\pm\infty$ as $x \to 0$. We will see presently that the power series method can be generalized to apply near the singular point $x = 0$ of (7.8), provided that $P(x)$ approaches infinity no more rapidly than $1/x$ (pole of order one), and $Q(x)$ no more rapidly than $1/x^2$ (pole of order two), as $x \to 0$. This is a way of saying that $P(x)$ and $Q(x)$ have only 'weak' singularities at $x = 0$.

To say this more precisely consider the following two functions:

$$p(x) := xP(x) \tag{7.10}$$

$$q(x) := x^2 Q(x) \tag{7.11}$$

Definition 7.1. Regular singular point
The singular point $x = 0$ of (7.8) is a **regular singular point** if the functions $p(x)$ and $q(x)$ are both analytic at $x = 0$. Otherwise it is an **irregular singular point**.

Remark 6: Consider the following:

$$\lim_{x\to 0} xP(x) = \lim_{x\to 0} p(x) = p_0$$
$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} q(x) = q_0$$

– If both of the above limits exist and are zero, then $x = 0$ may be an ordinary point.

– If both of the above limits exist and are nonzero, then $x = 0$ is a regular singular point.

– If either limit fails to exist or is infinite, then $x = 0$ is an irregular singular point.

Recall that we can write (7.8) in the equivalent form:

$$y'' + P(x)y' + Q(x)y = 0$$

where $B/A = P$ and $C/A = Q$. Multiplying by x^2 gives another equivalent form which will be convenient for our present analysis, namely,

$$x^2 y'' + xp(x)y' + q(x)y = 0 \tag{7.12}$$

Notice that if $p(x) = p_0$ and $q(x) = q_0$, i.e, are constants then (7.12) is a Cauchy-Euler equidimensional equation which can be solved by hand using the trial solution $y(x) = x^r$ where r is a root of the polynomial $r(r-1) + p_0 r + q_0 = 0$. In general if $x = 0$ is a regular singular point then p and q are analytic functions and one can write:

$$p(x) = p_0 + p_1 x + p_2 x^2 + \dots$$
$$q(x) = q_0 + q_1 x + q_2 x^2 + \dots$$

Using the Cauchy-Euler equation as motivation one can make an ansatz that there exists a solution to (7.8) of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n \tag{7.13}$$

which we call a **Frobenius series** solution. Note that this series is in general not a power series. For example, take $r = 1/2$, then (7.13) is not a power series. Finally, plugging the Frobenius series (7.13) into the differential equation (7.12) gives:

$$\begin{aligned}\left[r(r-1)c_0 x^r + (r+1)rc_1 x^{r+1} + \dots\right]& \\ + \left[p_0 x + p_1 x^2 + \dots\right] \cdot \left[rc_0 x^{r-1} + (r+1)c_1 x^r + \dots\right]& \\ + \left[q_0 + q_1 x + \dots\right] \cdot \left[c_0 x^r + c_1 x^{r+1} + \dots\right] = 0&\end{aligned} \tag{7.14}$$

Upon multiplying initial terms of the two products on the left-hand side here then collecting coefficients of x^r, we see that the lowest degree term in Eq. (7.14) is $c_0[r(r-1)+p_0 r+q_0]x^r$. If Eq. (7.14) is to be satisfied identically, then the coefficient of this term (as well as those of the higher-degree terms) must vanish. Since we can assume (without loss of generality) that $c_0 \neq 0$ it follows that r must satisfy the quadratic equation

$$r(r-1) + p_0 r + q_0 = 0. \tag{7.15}$$

We call (7.15) the **indicial equation** of the differential equation (7.12). This leads to the following important theorem.

Theorem 7.5. Frobenius series solutions
Suppose that $x = 0$ is a regular singular point of the equation

$$x^2 y'' + xp(x)y' + q(x)y = 0. \tag{7.16}$$

Let $\rho > 0$ be the minimum of the radii of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let r_1 and r_2 be the (real) roots, with $r_1 \geq r_2$, of the indicial equation $r(r-1) + p_0 r + q_0 = 0$. Then

a. For $x > 0$ (take absolute value for $x < 0$), there exists a solution of (7.16) of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0) \tag{7.17}$$

corresponding to the larger root r_1.

b. If $r_1 - r_2$ is neither zero nor a positive integer, then there exists a second linearly independent solution for $x > 0$ of the form

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad (b_0 \neq 0) \tag{7.18}$$

corresponding to the smaller root r_2.

Exceptional cases:

c. If $r_1 = r_2$, then (7.16) has two solutions y_1 and y_2 of the forms

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^2 \quad (a_0 \neq 0) \tag{7.19}$$

and

$$y_2(x) = y_1(x) \ln x + x^{r_1+1} \sum_{n=0}^{\infty} b_n x^n. \tag{7.20}$$

d. If $r_1 - r_2 = N$, a positive integer, then (7.16) has two solutions y_1 and y_2 of the forms

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0) \tag{7.21}$$

and

$$y_2(x) = Cy_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \tag{7.22}$$

In (7.22), $b_0 \neq 0$ but C may be either zero or nonzero, so the logarithmic term may or may not actually be present in this case. The radii of convergence of the power series of this theorem are all at least ρ. The coefficients in these series (and the constant C in (7.22)) may be determined by direct substitution of the series in the differential equation (7.16).

Remark 7: If the indicial equation has a pair of complex conjugate roots then there exists two distinct Frobenius series solutions.

Remark 8: The second linearly independent solution to (7.16) in the exceptional cases is derived using the method of reduction of order discussed earlier.

8 Hamiltonian mechanics:

9 Asymptotic methods: